Kartik Maths Real Analysis Paper-3

• Convergence of infinite series

Define Series or infinite Series —
The expression in the form $U_1 + U_2 + U_3 + U_4 + \dots + U_n + \dots$ of where $U_1, U_2, U_3 \dots U_n$ are in a sequence is known as an infinite Series.

finite Series :— A Series which contains finite no. of terms is known as finite Series.

eg., $U_1 + U_2 + U_3 + \dots + U_n$.

Special types of Series

convergent Series :— An infinite Series $U_1 + U_2 + U_3 + \dots + U_n + \dots$ is said to be convergent if $S_n \to S$ as $n \to \infty$ where S_n is the sum of n terms in ΣU_n and S is a finite ~~Series~~ unit

i.e. $S_n = S$

$n \to \infty$

Q. Test the convergence of the Series.

$$1 + \frac{1}{3^1} + \frac{1}{3^2} + \frac{1}{3^3} + \dots + \frac{1}{3^{n-1}} \dots \infty$$

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Soln:- Let S_n be the sum of n terms of given series

$\therefore$ the series is in G.P

$\therefore a = 1 \;\&\; r = \frac{1}{3} < 1$

$\therefore S_n = \frac{a(1 - r^n)}{(1 - r)}$

$= \frac{1\left\{1 - \left(\frac{1}{3}\right)^n\right\}}{\left(1 - \frac{1}{3}\right)}$

$= \frac{1 - \left(\frac{1}{3}\right)^n}{2/3}$

$= \frac{3}{2}\left(1 - \frac{1}{3^n}\right)$

$\Rightarrow S_n = \frac{3}{2}\left(1 - \frac{1}{3^n}\right)$

$\therefore \lim_{n \to \infty} S_n = \frac{3}{2}\left(1 - \frac{1}{\infty}\right)$

$= \frac{3}{2}(1 - 0)$

$= 3/2$

which is finite

Hence, the given series is convergent

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Q.) prove that the series $a + ar + ar^2 + ar^3 \cdots$ to ∞ which is series in G.P is convergent when commen ratio $|r| < 1$, divergent when $r \geq 1$ and oscillating when $r < -1$

Soln:- Let S_n be the sum of the given Series as

$$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1} \quad \text{— (i)}$$

multipling both sides by r we get

$$r \times S_n = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} + ar^n \quad \text{— (2)}$$

Subtracting (2) from (i) we get

$$S_n - rS_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} - ar - ar^2 - ar^3 - \cdots - ar^{n-1} - ar^n$$

$$S_n - rS_n = a - ar^n$$

$$\Rightarrow S_n(1-r) = a - ar^n$$

$$\Rightarrow S_n = \frac{a}{(1-r)} - \frac{ar^n}{(1-r)} \quad \text{— (3)}$$

Case-1 When $|r| < 1$

$n \to \infty$ then $r^n \to 0$

$\therefore$ from (3)

$$S_n = \frac{a}{(1-r)}$$

$$\left[\begin{array}{l} |x| < a \\ \Rightarrow -a < x < a \end{array} \right.$$

which is finite

$\therefore$ The series is convergent.

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Case - II

When $r = 1$

$S_n = a + a + a + \dots + \infty$

$= \infty$ as $n \to \infty$

$\Rightarrow S_n \to \infty$ on $n \to \infty$

The series is divergent

Case III

When $r > 1$

then $S_n > a_n$

$\Rightarrow S_n \to a_n$ as $n \to \infty$

$\therefore$ the series is divergent.

Case IV

When $r = -1$

$S_n = 0$ or 1

according as n is even or odd.

the series is oscillating

Case V

When $r < -1$

then $S_n \to +\infty$ or $-\infty$

according as n is odd or even

$\therefore$ the series is oscillating

proved

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Q.) Define oscillating series if the sum of n term $\to \infty$ limit, whether finite or infinite the series is said to be an oscillating series.

Soln :— The series $2-2+2-2+2-2\cdots$

Here, $\lim_{n\to\infty} S_n = 0$ or 2

according to as n even or odd

Thus the given series oscillator between certain limit

Hence

The series is an oscillating series.

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9nP.

Theorem (2): State and prove the cauchy's genral principle of convergence of series.

Statement :- A necessary and Sufficient condition for a series Σu_n to converge is that to each $\varepsilon > 0$ there exist a positive integer in such that

$$|u_{m+1} + u_{m+2} + \cdots + u_n| < \varepsilon.$$

whenever $n > m$

proof :→

The series $|u_m|$ converges if & only if the sequence S_n of its partial sums converges. By cauchy's genral principle of convergenie for sequences, a necessary and Sufficient condition for the convergenic of sequence $\{S_n\}$ is then given for $\varepsilon > 0$ there exist a +ve integer m such that

$$|S_n - S_m| < \varepsilon, \quad n > m$$

i.e $|u_{m+1} + u_{m+2} + \cdots + u_n| < \varepsilon, \quad n > m$

proved.

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9nf

Theorem :- (3) Prove that an infinite series ΣV_n is convergent then $V_n \to 0$ as $n \to \infty$

Proof :- let ΣV_n is convergent

Also,

let $S_n = V_1 + V_2 + V_3 + \cdots + V_n$

and

let $S_n \to S$ as $n \to \infty$.

Here,

$$S_n = V_1 + V_2 + V_3 + \cdots + V_{n-1} + V_n$$

$$S_{n-1} = V_1 + V_2 + V_3 + \cdots + V_{n-1}$$

$$S_n - S_{n-1} = (V_1 + V_2 + V_3 + \cdots + V_{n-1} + V_n) - (V_1 + V_2 + V_3 + \cdots + V_{n-1})$$

$$S_n - S_{n-1} = V_n$$

Taking limit on both sides $n \to \infty$

$$\lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = \lim_{n\to\infty} V_n$$

$$\Rightarrow S - S = \lim_{n\to\infty} V_n$$

$$\Rightarrow \lim_{n\to\infty} V_n = 0$$

$$\Rightarrow V_n \to 0 \text{ as } n \to \infty$$

proved

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V.V.I

Theorem (4) :- State and Prove P-Series test or auxiliary Series test.

Statement :- The infinite series $\Sigma \frac{1}{n^p}$ i.e

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \cdots + \infty$$

(i) convergent if $p > 1$

(ii) Divergent if $p \leq 1$

OR Test the convergent of the Series $\Sigma \frac{1}{n^p}$

Proof :- Case - I if $p > 1$

$$\frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \cdots + \infty$$

Here, we have,

$$\frac{1}{1^p} = 1$$

$$\frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^p} + \frac{1}{2^p} = \frac{2}{2^p}$$

$$\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} < \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}$$

$$= \frac{4}{4^p} = \left(\frac{2}{2^p}\right)^2$$

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Adding Sidewise We get-

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} + \cdots \infty$$

$$< 1 + \frac{2}{2^p} + \left(\frac{2}{2^p}\right)^2 + \cdots + \infty$$

$$= \frac{1}{1 - \frac{2}{2^p}}$$ which is finite.

∴ The Series is convergent if $p > 1$

Case II

if $p = 1$

$$\left(1 + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots \infty$$

Here we have

$$1 + \frac{1}{2} > \frac{1}{2} + \frac{1}{2} = 1$$

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{2}{4}$$

$$= \frac{1}{2}$$

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$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$

$= \frac{4}{8} = \frac{1}{2}$

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- - - - - - - - - So on.

adding sidewise we get,

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots + \infty$

$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \infty$

which is not finite

∴ The series is divergent if $p = 1$

case-III if $p < 1$

$\frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \cdots + \infty$

$1^p = 1, \quad \frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{4} + \frac{1}{4} = 2/4 = 1/2,$

Adding sidewise, we get

$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} + \cdots + \infty < 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots \infty$

which is infinite

∴ The series is divergent if $p < 1$

Proved

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Theorem ⑤ : State and prove comparison test

Statement :- If $\lim_{n\to\infty} \frac{u_n}{v_n}$ = a finite and non-zero number l, then Σu_n and Σv_n we either both convergent or both divergent.

Proof :- Let $\lim_{n\to\infty} \frac{u_n}{v_n} = l$

Here, l is finite and non-zero

Now, for every $\epsilon > 0$ there exists a positive integer m such that

$$\left|\frac{u_n}{v_n} - l\right| < \epsilon \quad \text{as} \quad n \geq m$$

[By defination of limit]

$$\Rightarrow l - \epsilon < \frac{u_n}{v_n} < l + \epsilon \quad \text{as} \quad n \geq m \quad [\because |f(x)-l| < \epsilon]$$

$$\Rightarrow v_n(l-\epsilon) < u_n < v_n(l+\epsilon) \quad \left[\lim_{x\to a} f(x) = l\right]$$

Hence if Σv_n is convergent then $\Sigma v_n(l+\epsilon)$ is also convergent
$\Rightarrow \Sigma u_n$ is convergent

Also, if Σv_n is divergent then $\Sigma v_n(l-\epsilon)$ is divergent
$\Rightarrow \Sigma u_n$ is divergent Proved

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n^4 n^4

Q.1.) Test the convergent of the series

$$1 + \frac{1+2}{1+2^2} + \frac{1+3}{1+3^2} + \cdots + \frac{1+n}{1+n^2} + \cdots + \text{to } \infty$$

or test the convergence of the series $\sum_{n=1}^{\infty} \frac{1+n}{1+n^2}$

Soln :— Here, $u_n = \frac{1+n}{1+n^2}$

$\therefore$ The auxiliary series be

$$\sum v_n = \sum \frac{n}{n^2}$$

$$= \sum \frac{1}{n}$$

$$\therefore v_n = \frac{1}{n}$$

Now,

$$\lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty} \frac{\left(\frac{1+n}{1+n^2}\right)}{1/n}$$

$$= \lim_{n \to \infty} \frac{(1+n)}{1+n^2} \cdot n$$

$$= \lim_{n \to \infty} \frac{n\left(\frac{1}{n}+1\right) \cdot n}{n^2\left(\frac{1}{n^2}+1\right)}$$ ⇒, We have

$$= \frac{0+1}{0+1}$$

$= 1$, which is finite no. is a geometric

$1 + \epsilon < 1$

$\therefore \sum (1+\epsilon)^n$ convergent

Here, the Series $\sum u_n$ is also convergent

Again,

If $l > 1$

$1 - \epsilon > 1$

then, the second inequality from (i) we get

$u_n^{1/n} > 1 - \epsilon, \quad n \geq m$

$\Rightarrow u_n > (1-\epsilon)^n, \quad n \geq m$

Hence, the series $\sum (1-\epsilon)^n$ is geometric series with common ratio

$1 - \epsilon > 1$

$\therefore \sum (1-\epsilon)^n$ is divergent

Hence the Series $\sum u_n$ is also divergent

Similarly if $l = 1$

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Q.) Test the convergence of the series:-

$$\frac{1}{4} + \frac{1\cdot3}{4\cdot7} + \frac{1\cdot3\cdot5}{4\cdot7\cdot10} + \frac{1\cdot3\cdot5\cdot7}{4\cdot7\cdot10\cdot13} + \cdots$$

Soln:- Let n^{th} term is denoted by u_n

$$\therefore u_n = \frac{1\cdot3\cdot5\cdot7\cdots(2n-1)}{4\cdot7\cdot10\cdot13\cdots(3n+1)}$$

$$u_{n+1} = \frac{1\cdot3\cdot5\cdot7\cdots(2n-1)\cdot\{2(n+1)-1\}}{4\cdot7\cdot10\cdot13\cdots(3n+1)\{3(n+1)+1\}}$$

$$= \frac{1\cdot3\cdot5\cdot7\cdots(2n-1)(2n+1)}{4\cdot7\cdot10\cdot13\cdots(3n+1)(3n+4)}$$

Now,

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{1\cdot3\cdot5\cdot7\cdots(2n-1)}{4\cdot7\cdot10\cdot13\cdots(3n+1)} \cdot \frac{4\cdot7\cdot10\cdot13\cdots(3n+1)(3n+4)}{1\cdot3\cdot5\cdot1\cdots(2n-1)(2n+1)}$$

$$= \lim_{n\to\infty} \frac{(3n+4)}{(2n+1)} = \lim_{n\to\infty} \frac{n(3+4/n)}{n(2+1/n)}$$

$$= \frac{3+0}{2+0} = 3/2 > 1$$

$\therefore$ By ratio test $\sum u_n$ is convergent

Ratio Test

If $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} > 1$ Convergence

$\lim_{n\to\infty} < 1$ divergent.

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Q.) Determine the convergence ...

$$2x + \frac{3x^2}{8} + \frac{4x^3}{27} + \cdots + \frac{(n+1)}{n^3} \cdot x^n + \cdots \text{to}\infty$$

Soln:-

Let the n^{th} term is denoted by u_n as

$$u_n = \frac{(n+1)\, x^n}{n^3}$$

$$u_{n+1} = \frac{\{(n+1)+1\}\, x^{n+1}}{(n+1)^3} = \frac{(n+2)\, x^{n+1}}{(n+1)^3}$$

Now $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{\frac{(n+1)}{n^3}}{\frac{(n+2)}{(n+1)^3}} \times \frac{x^n}{x^{n+1}}$

$$= \lim_{n\to\infty} \frac{(n+1) \times (n+1)^3}{n^3 \times (n+2)} \times \frac{x^n}{x^{n+1}}$$

$$= \lim_{n\to\infty} \frac{n^4\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^3}{n^4\left(1+\frac{2}{n}\right)} \cdot \frac{1}{x}$$

$$= \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x}$$

If $x > 1$ then $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} < 1$ $\therefore \sum u_n$ is divergent

If $x < 1$ then $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} > 1$ $\therefore \sum u_n$ is convergent

If $x = 1$ then ratio test fail $\therefore u_n = \frac{n+1}{n^3}$

Here, $v_n = \frac{n}{n^3}$, $\lim_{n\to\infty} \frac{u_n}{v_n}$.

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V.V.I.

Q. Discuss the convergence of the series

$$\frac{x}{1.2} + \frac{x^2}{3.4} + \frac{x^3}{5.6} + \frac{x^4}{7.8} - - - \quad (x>0)$$

V.V.I.

Q.) Test the series

$$\sum_{n=1}^{\infty} \frac{x^n}{1+n^2} \quad \text{for all } x>0$$

V.V.I.

Q Test the convergence of the series whose general term is $\frac{n^2}{n^2+1} x^{n-1}$ $(x>0)$

V.V.I.

Q n^{th} term $\frac{\sqrt{n^2+1}}{\sqrt{n^2+1}}, x^n$ $(n>0)$

Q. Discuss the convergency & divergence of the following series:

(i) $1 + \frac{2}{3^2} + \frac{2.4}{3^2.5^2} + \frac{2.4.6}{3^2.5^2.7^2} + \cdots$ to ∞

(ii) $1 + \frac{2^{-p}}{\lfloor 2} + \frac{3^{-p}}{\lfloor 3} + \cdots \frac{n^{-p}}{\lfloor n} + \cdots$ to ∞

(iii) $\frac{3}{4} + \frac{1}{2}x + \frac{1}{12}x^2 + \cdots + \frac{2n-5}{n^3-5^n} x^{n-1} + \cdots$ to ∞

(iv) $1 + \frac{x}{3} + \frac{x^2}{5} + \cdots$ to ∞ $[x]>0$

(v) $\frac{x}{1.2} + \frac{x^2}{2.3} + \frac{x^3}{3.4} + \frac{x^4}{4.5} + \cdots$ to ∞ $(x>0)$

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V.V.I

Ex. Discuss the convergence of the series

$$\frac{x}{1.2} + \frac{x^2}{3.4} + \frac{x^3}{5.6} + \frac{x^4}{7.8} + \cdots \quad (x>0)$$

Soln:-

Let U_n be the n^{th} term

$$\therefore U_n = \frac{x^n}{(2n-1)\cdot 2n}$$

$$\therefore U_{n+1} = \frac{x^{n+1}}{(2n+1)(2n+2)}$$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \lim_{n\to\infty} \frac{x^n}{(2n-1)\cdot 2n} \cdot \frac{(2n+1)(2n+2)}{x^{n+1}}$$

$$= \lim_{n\to\infty} \frac{x^n}{n\left(2-\frac{1}{n}\right)\cdot 2n} \cdot \frac{n\left(2+\frac{1}{n}\right)\cdot n\left(2+\frac{2}{n}\right)}{x^n \cdot x}$$

$$= \frac{(2+0)\cdot(2+0)}{(2-0)\cdot 2} \cdot \frac{1}{x}$$

$$= \frac{4}{2.2}\cdot\frac{1}{x} = \frac{1}{x}$$

If $x > 1$ then $\lim_{n\to\infty} \frac{U_n}{U_{n+1}} < 1$

$\therefore \sum U_n$ is divergent

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If $x < 1$, then $\lim_{n \to \infty} \frac{u_n}{u_{n+1}} > 1$

$\therefore \sum u_n$ is convergent

If $n = 1$ then

$$u_n = \frac{1}{(2n-1) \cdot 2n}$$

$$\therefore v_n = \frac{1}{n \cdot n} = \frac{1}{n^2}$$

:

$\lim_{n \to \infty} \frac{u_n}{v_n}$

$$= \lim_{n \to \infty} \frac{1}{(2n-1) \cdot 2n} \cdot n^2$$

$$= \lim_{n \to \infty} \frac{n^2}{n(2 - \frac{1}{n}) \cdot 2n}$$

$$= \frac{1}{2(2-0)} = \frac{1}{4}$$

$= \frac{1}{4}$, which is finite & non-zero

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V.V.I

Q. Discuss the convergence of the series

$$\sum_{n=1}^{\infty} \frac{x^n}{(n^2+1)}$$

Soln :-

Let u_n be the nth term

$$u_n = \frac{x^n}{n^2+1}$$

$$\therefore u_{n+1} = \frac{x^{n+1}}{(n+1)^2+1}$$

$$= \frac{x^{n+1}}{n^2+2n+1+1}$$

$$= \frac{x^{n+1}}{n^2+2n+2}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{x^n}{(n^2+1)} \times \frac{(n^2+2n+2)}{x^{n+1}}$$

$$= \lim_{n\to\infty} \frac{x^n}{n^2\left(1+\frac{1}{n^2}\right)} \times \frac{n^2\left(1+\frac{2}{n}+\frac{2}{n^2}\right)}{x^n \cdot x}$$

$$= \frac{(1+0+0)}{(1+0)} \cdot \frac{1}{x}$$

$$= 1/x$$

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If $x < 1$ then $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} > 1$

$\therefore \sum u_n$ is convergent

If $x > 1$ then $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} < 1$

$\therefore \sum u_n$ is divergent

If $x = 1$ then

$$u_n = \frac{1}{n^2+1}$$

$$\therefore v_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{(n^2+1)} \cdot \frac{n^2}{1}$$

$$= \lim_{n\to\infty} \frac{1}{n^2\left(1+\frac{1}{n^2}\right)} \cdot n^2$$

$$= \frac{1}{(1+0)} = \frac{1}{1}$$

$= 1$, which is finite & non-zero

If $x = 1$, then ratio test will be failed.

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Ex Discuss the convergence of the series for real value of x

$$\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \cdots \text{ to } \infty.$$

Soln:— Let u_n be the n^{th} term

$$\therefore u_n = \frac{x^{2n-2}}{(n+1)\sqrt{n}}$$

$$u_{n+1} = \frac{x^{2(n+1)-2}}{\{(n+1)+1\}\sqrt{n+1}}$$

$$= \frac{x^{2n}}{(n+2)\sqrt{n+1}}$$

Now,

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{x^{2n-2}}{(n+1)\sqrt{n}} \times \frac{(n+2)\sqrt{n+1}}{x^{2n}}$$

$$= \lim_{n\to\infty} \frac{x^{2n} \cdot x^{-2}}{x\left(1+\frac{1}{n}\right)\cdot\sqrt{n}} \times \frac{x\left(1+\frac{2}{n}\right)\cdot\sqrt{n}\sqrt{1+\frac{1}{n}}}{x^{2n}}$$

$$= \frac{1}{x^2} \frac{(1+0)\sqrt{1+0}}{1+0}$$

$$= \lim \frac{1}{x^2}$$

If $x^2 < 1 \Rightarrow -1 < x < 1$

then, $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} > 1$

$\therefore \sum u_n$ is convergent

If $x^2 > 1 \Rightarrow x > 1$ & $x < -1$

then, $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} < 1$

$\therefore \sum u_n$ is divergent.

If $x^2 = 1 \Rightarrow x = \pm 1$

$u_n = \frac{1}{(n+1)\sqrt{n}}$

$\therefore v_n = \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{(n+1)\sqrt{n}} \cdot n\sqrt{n}$$

$$= \lim_{n\to\infty} \frac{n}{n(1+\frac{1}{n})}$$

$$= \frac{1}{1+0} = 1$$

which is finite or non-zero.

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Q. Discuss the convergency or divergency of the given series

$$1 + \frac{2}{3^2} + \frac{2.4}{3^2.5^2} + \frac{2.4.6}{3^2.5^2.7^2} + \cdots \text{ to } \infty.$$

Soln:-

Let u_n be the n^{th} term

$$\therefore u_n = \frac{2.4.6 \ldots 2n}{3^2.5^2.7^2 \ldots (2n+1)^2}$$

$$\therefore u_{n+1} = \frac{2.4.6 \ldots 2n.[2(n+1)]}{3^2.5^2.7^2 \ldots (2n+1)^2 \{2(n+1)+1\}^2}$$

$$= \frac{2.4.6 \ldots 2n(2n+2)}{3^2.5^2.7^2 \ldots (2n+1)^2 (2n+3)^2}$$

Now, $\lim_{n \to \infty} \frac{u_n}{u_{n+1}}$

$$= \lim_{n \to \infty} \frac{2.4.6 \ldots 2n}{3^2.5^2.7^2 \ldots (2n+1)^2} \times \frac{3^2.5^2.7^2 \ldots (2n+1)^2 (2n+3)^2}{(2.4.6 \ldots 2n)(2n+2)}$$

$$= \lim_{n \to \infty} \frac{(2n+3)^2}{(2n+2)}$$

$$= \infty$$

$\therefore \Sigma u_n$ is divergent

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V.V.I.

Q. Discuss the convergency & divergency of given series $\sum \frac{\lfloor n}{n^n}$

Soln:—

Let u_n be the n^{th} term of the series

$$\therefore u_n = \frac{\lfloor n}{n^n}$$

$$u_{n+1} = \frac{\lfloor n+1}{(n+1)^{n+1}}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{\lfloor n}{n^n} \times \frac{(n+1)^{n+1}}{\lfloor n+1}$$

$$= \lim_{n\to\infty} \frac{\lfloor n}{n^n} \times \frac{(n+1)^{n+1}}{(n+1)\lfloor n}$$

$$= \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)^n}$$

$$= \lim_{n\to\infty} \frac{\lfloor n}{n^n} \times \frac{(n+1)^n \cdot (n+1)}{(n+1)\lfloor n}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$$

$$= e > 1 \quad [2 < e < 3]$$

which is finite or non-zero

∴ by ratio test

$\sum u_n$ is convergent

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V.V.I

R.R.I.

Theorem :- State and prove Raabe's Test.

Statement :- The series ΣU_n of positive terms is convergent if

$$\lim_{n\to\infty} \left\{ n\left(\frac{U_n}{U_{n+1}} - 1\right)\right\} > 1$$

or,

divergent if $\lim_{n\to\infty} \left\{ n\left(\frac{U_n}{U_{n+1}} - 1\right)\right\} < 1$

Proof :-

Let the given series be

$$U_1 + U_2 + U_3 + \cdots + U_n + U_{n+1} + \cdots \text{ to } \infty$$

Let $\Sigma V_n = \Sigma \frac{1}{n^p}$ is auxiliary series.

Here $V_n = \frac{1}{n^p}$

$$V_{n+1} = \frac{1}{(n+1)^p}$$

$$\therefore \frac{V_n}{V_{n+1}} = \frac{1}{n^p} \times \frac{(n+1)^p}{1}$$

$$= \left(\frac{n+1}{n}\right)^p$$

$$= \left(1 + \frac{1}{n}\right)^p$$

$\therefore \Sigma V_n = \Sigma \frac{1}{n^p}$ is convergent or divergent according as $p > 1$ or $p < 1$

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$(1+x)^n = 1 + x.n + \frac{n(n-1)}{\lfloor 2}x^2 + \frac{n(n-1)(n-2)}{\lfloor 3}x^3$

Hence by second comparison test, the series Σu_n is convergent or divergent according as

$$\frac{u_n}{u_{n+1}} > \text{or} < \frac{v_n}{v_{n+1}}$$

$$\Rightarrow \frac{u_n}{u_{n+1}} > \text{or} < \left(1+\frac{1}{n}\right)^p$$

$$\Rightarrow \frac{u_n}{u_{n+1}} > \text{or} < 1 + p.\frac{1}{n} + \frac{p(p-1)}{\lfloor 2}.\frac{1}{n^2} + \cdots$$

(By binomial theorem)

$$(1+x)^n = 1 + x.n + \frac{n(n-1)}{\lfloor 2}.x^2 + \frac{n(n-1)(n-1)}{\lfloor 3}x^3$$

$$\Rightarrow \frac{u_n}{u_{n+1}} - 1 > \text{or} < p.\frac{1}{n} + \frac{p(p-1)}{\lfloor 2\, n^2} + \cdots$$

$$\Rightarrow \frac{u_n}{u_{n+1}} - 1 > \text{or} < \frac{1}{n}\left[p + \frac{p(p-1)}{\lfloor 2}.\frac{1}{n} + \cdots\right]$$

$$\Rightarrow n\left(\frac{u_n}{u_{n+1}} - 1\right) > \text{or} < p + \frac{p(p-1)}{\lfloor 2}.\frac{1}{n} + \cdots$$

Taking limit on all sided $n \to \infty$ we get

$$\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} > \text{or} < p$$

$$\Rightarrow \lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} > \text{or} < 1$$

Proved

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Ex 6.2

① Given $1^2 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \cdots$

Soln:—

Let u_n be the n^{th} term

$\therefore u_n = n^2 x^{n-1}$

$\therefore u_{n+1} = (n+1)^2 \cdot x^n$

Now, $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{n^2 x^{n-1}}{(n+1)^2 \cdot x^n}$

$= \lim_{n\to\infty} \frac{n^2 \cdot x^n \cdot x^{-1}}{n^2\left(1+\frac{1}{n}\right)^2 \cdot x^n}$

$\therefore = \frac{1}{(1+0)} \cdot \frac{1}{x} = \frac{1}{x}$

$\therefore$ By D. Alembert's Ratio test we have

If $x < 1$ then $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} > 1$

$\therefore \sum u_n$ is convergent

If $x > 1$ then $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} < 1$

$\therefore \sum u_n$ is divergent

If $x = 1$, then this test fails

$u_n = n^2$

$\lim_{n\to\infty} u_n = \infty$

$\therefore \sum u_n$ is divergent

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(3) Given $1 + \frac{x}{3} + \frac{x^2}{5} + \frac{x^3}{7} + \cdots$

Soln:- Let u_n ~~is~~ be the n^{th} term

$$\therefore u_n = \frac{x^{n-1}}{(2n-1)}$$

$$\therefore u_{n+1} = \frac{x^n}{\{2(n+1)-1\}}$$

$$= \frac{x^n}{(2n+1)}$$

Now, $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{x^{n-1}}{(2n-1)} \times \frac{(2n+1)}{x^n}$

$$= \lim_{n\to\infty} \frac{x^n \cdot x^{-1}}{x(2n-\frac{1}{n})} \times \frac{x(2+\frac{1}{n})}{x^n}$$

$$= \frac{x^{-1}(2+0)}{(2-0)} = \frac{1}{x}$$

If $x < 1$ then, $\frac{u_n}{u_{n+1}} > 1$

$\therefore \sum u_n$ is convergent

If $x > 1$ then $\frac{u_n}{u_{n+1}} < 1$

$\therefore \sum u_n$ is divergent

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If $n = 1$ then

$$u_n = \frac{1}{(2n-1)}$$

$$= \frac{1}{(2-1)} \qquad v_n = \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{(2n-1)} \cdot n$$

$$= \lim_{n\to\infty} \frac{n}{n\left(2 - \frac{1}{n}\right)}$$

$$= \frac{1}{(2-0)}$$

$= \frac{1}{2}$, which is finite or non zero.

By the comparison test

$$v_n = \frac{1}{n}$$

$\therefore p = 1$

$\therefore \Sigma u_n$ is divergent

Σv_n is convergent

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(5) $\frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{3}{4}x^4 + \frac{4}{5}x^5 + \dots$

Soln:-

Let U_n be the n^{th} term

$$U_n = \frac{n x^{n+1}}{(n+1)}$$

$$U_{n+1} = \frac{(n+1)\, x^{(n+1)+1}}{(n+1)+1} \Rightarrow \frac{(n+1)\, x^{n+2}}{(n+2)}$$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \lim_{n\to\infty} \frac{n x^{n+1}}{(n+1)} \times \frac{(n+2)}{(n+1)\, x^{n+2}}$$

$$= \lim_{n\to\infty} \frac{x\, x^n \cdot x}{x(1+\frac{1}{n})} \cdot \frac{x(1+2/n)}{x(1+\frac{1}{n})\, x^n \cdot x^2}$$

$$= \frac{(1+0)}{(1+0)} \cdot \frac{1}{x} = \frac{1}{x}$$

which is finite & non zero.

If $x < 1$, then $\frac{U_n}{U_{n+1}} > 1$

$\therefore \sum U_n$ is convergent

If $x > 1$ then $\frac{U_n}{U_{n+1}} < 1$

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$\therefore \Sigma u_n$ is divergent

If $n = 1$ then,

$$u_n = \frac{1}{n+1}$$

$$v_n = \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{(n+1)} \cdot \frac{n}{1}$$

$$= \lim_{n\to\infty} \frac{1}{n\left(1+\frac{1}{n}\right)} \cdot \frac{n}{1}$$

$$= \frac{1}{(1+0)} = 1$$

Which is finite or non zero

By D. Alembert's Test

$k > 1$ then

$$\frac{u_n}{u_{n+1}} < 1$$

$\therefore \Sigma u_n$ is divergent

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V.V.I

Theorem: State and prove Logarithmic Ratio test or logarithmic test.

Statement :-

The series Σu_n be positive terms is convergent if

$$\lim_{n \to \infty} \left(n \log \frac{u_n}{u_{n+1}} \right) > 1$$

or divergent if

$$\lim_{n \to \infty} \left\{ n \log \frac{u_n}{u_{n+1}} \right\} < 1$$

Proof :-

Let the series Σu_n is given as

$$u_1 + u_2 + u_3 + \cdots + u_n + u_{n+1} + \cdots \text{ to } \infty$$

Let the auxiliary series be

$$\Sigma v_n = \Sigma \frac{1}{n^p}$$

$$\therefore v_n = \frac{1}{n^p}$$

& $v_{n+1} = \frac{1}{(n+1)^p}$

$$\therefore \frac{v_n}{v_{n+1}} = \frac{1}{n^p} \times (n+1)^p = \left(\frac{n+1}{n}\right)^p$$

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We know that Σu_n is convergent or divergent according as $p>1$ or $p<1$

$\therefore$ By the Secondal comparison test the Series Σu_n is convergent or divergent according as

$$\frac{u_n}{u_{n+1}} > \text{or} < \frac{v_n}{v_{n+1}}$$

$$\Rightarrow \frac{u_n}{u_{n+1}} > \text{or} < \left(\frac{n+1}{n}\right)^p$$

taking log both sides

$$\log \frac{u_n}{u_{n+1}} > \text{or} < p \log \left(\frac{n+1}{n}\right)$$

$$\Rightarrow \log \frac{u_n}{u_{n+1}} > \text{or} < p \log \left(1+\frac{1}{n}\right)$$

$$\Rightarrow \log \frac{u_n}{u_{n+1}} > \text{or} < p\left[\frac{1}{n} - \frac{1}{2n^2} + \cdots\right]$$

$$\left\{\because \log(1+x) = x + \frac{x^2}{2} + \frac{x^3}{3}\right)$$

$$\Rightarrow \log \frac{u_n}{u_{n+1}} > \text{or} < \frac{p}{n}\left[1 - \frac{1}{2n} + \cdots\right]$$

$$\Rightarrow n \log \frac{u_n}{u_{n+1}} > \text{or} < p\left(1 - \frac{1}{2n} + \cdots\right)$$

$$\Rightarrow \lim_{n\to\infty} \left\{ n \log \frac{u_n}{u_{n+1}} \right\} > \text{or} < p(1-0)$$

$$\Rightarrow \lim_{n\to\infty} \left\{ n \log \frac{u_n}{u_{n+1}} \right\} > \text{or} < 1$$

Proved.

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Ans.

8.) Theorem :- Prove that the series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$

i.e $\frac{1}{2(\log 2)^p} + \frac{1}{3(\log 3)^p} + \cdots + \frac{1}{n(\log n)^p} + \cdots$ to ∞

is convergent if $p > 1$ and diverges if $p < 1$

Proof :- Let $f(n)$ be the nth term of given series

$\therefore f(n) = \frac{1}{n(\log n)^p}$

Here, $n(\log n)^p$ is positive and increasing as $p > 0$

If $p \to 0$ $p < 0$,

then $\frac{n}{(\log n)^p}$ is positive & increasing

Let a be a +ve number greater than unity

$\therefore a^n f(a^n) = a^n \cdot \frac{1}{a^n (\log a^n)^p}$

$= \frac{1}{(n \log a)^p}$

$= \frac{1}{n^p \cdot (\log a)^p}$

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6 Aug 2001

$$\Rightarrow \sum_{n=2}^{\infty} a_n f(a_n) = \sum_{n=2}^{\infty} \frac{1}{n^p (\log a)^p}$$

$$= \frac{1}{(\log a)^p} \sum_{n=2}^{\infty} \frac{1}{n^p}$$

there the series $\sum_{n=2}^{\infty} \frac{1}{n^p}$ is convergent if and divergent if $p \leq 1$

$\therefore \sum_{n=2}^{\infty} a_n \cdot f(a_n)$ is convergent

if $p > 1$ and divergent if $p \leq 1$

Hence the given series

$$\sum_{n=2}^{\infty} f(n) = \sum_{n=2}^{\infty} \frac{1}{n (\log n)^p}$$ is convergent

if $p > 1$ and divergent if $p < 1$

proved

8. Theorem :- State and prove De morgen's and Bertrand's Test.

Statement :- If Σu_n is a series then it is convergent or divergent o/c as

$$\lim_{n\to\infty}\left[p\log n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right]\log n > 1 \text{ or } \leq 1$$

Proof :- Let the series Σu_n is given as

$$u_1 + u_2 + u_3 + \cdots + u_n + u_{n+1} + \cdots \text{ to } \infty$$

We compare the series Σu_n with the auxiliary series

$$1 + \frac{1}{2(\log 2)^p} + \frac{1}{3(\log 3)^p} + \cdots + \frac{1}{n(\log n)^p} + \cdots \text{ to } \infty$$

Let $v_n = \frac{1}{n(\log n)^p}$

$$\therefore v_{n+1} = \frac{1}{(n+1)\{\log(n+1)\}^p}$$

$$\therefore \frac{v_n}{v_{n+1}} = \frac{1}{n(\log n)^p} \times (n+1)\{\log(n+1)\}^p$$

$$= \left(\frac{n+1}{n}\right)\left\{\frac{\log(1+n)}{\log n}\right\}^p$$

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Now, we know that Σu_n is convergent or divergent a/o as $p > 1$ or $p < 1$, Hence by 2nd comparison test. Σu_n is convergent or divergent

a/c $\frac{u_n}{u_{n+1}} > \text{or} < \frac{v_n}{v_{n+1}}$

$\Rightarrow \frac{u_n}{u_{n+1}} > \text{or} < \left(\frac{n+1}{n}\right)\left\{\frac{\log(1+n)}{\log n}\right\}^p$

$\Rightarrow \frac{u_n}{u_{n+1}} > \text{or} < \left(\frac{n+1}{n}\right)\left\{\frac{\log\left\{n\left(1+\frac{1}{n}\right)\right\}}{\log n}\right\}^p$

$\Rightarrow \frac{u_n}{u_{n+1}} > \text{or} < \left(1+\frac{1}{n}\right)\left[\frac{\log n + \log\left(1+\frac{1}{n}\right)}{\log n}\right]^p$

$\Rightarrow \frac{u_n}{u_{n+1}} > \text{or} < \left(1+\frac{1}{n}\right)\left[\frac{\log n}{\log n} + \frac{\log\left(1+\frac{1}{n}\right)}{\log n}\right]^p$

$\Rightarrow \frac{u_n}{u_{n+1}} > \text{or} < \left(1+\frac{1}{n}\right)\left[\frac{1 + \frac{1}{n} - \frac{1}{2n^2} + \cdots}{\log n}\right]^p$

$\left[\because \log(1+x) = x - \frac{1}{2}x^2 + \cdots\right]$

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$$\Rightarrow \frac{u_n}{u_{n+1}} > \text{or} < \left(1+\frac{1}{n}\right)\left[1+\left\{\frac{1}{n(\log n)} - \frac{1}{2n^2(\log n)^2}\right\}\right]^p$$

$$\Rightarrow \frac{u_n}{u_{n+1}} > \text{or} < \left(1+\frac{1}{n}\right)\left[1+\frac{p}{n(\log n)} - \frac{p}{2n^2(\log n)} + \cdots\right]$$

[By binomial theorem
$(1+x)^p = 1+px+\cdots$]

$$\Rightarrow \frac{u_n}{u_{n+1}} > \text{or} < 1+\frac{1}{n}+\frac{p}{n(\log n)}+\frac{p}{n^2(\log n)}-\frac{p}{2n^2\log n}$$

$$\Rightarrow \frac{u_n}{u_{n+1}} - 1 > \text{or} < \frac{1}{n}+\frac{p}{n(\log n)}+\frac{p}{2n^2(\log n)}+\cdots$$

$$\Rightarrow n\left(\frac{u_n}{u_{n+1}} - 1\right) > \text{or} < 1+\frac{p}{\log n}+\frac{p}{2n\log n}$$

$$\Rightarrow \left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\} > \text{or} < \frac{p}{(\log n)}+\frac{p}{2n\log n}+\cdots$$

$$\Rightarrow \left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\}\log n > \text{or} < p+\frac{p}{2n}$$

$$\Rightarrow \lim_{n\to\infty}\left[\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right)\log n\right] > \text{or} < p$$

$$\Rightarrow \lim_{n\to\infty}\left[\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\}\log n\right] > \text{or} < 1$$

Proved

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V.V.I

Q/ State and Prove Kummer's Test.

Statement :- Let Σu_n and $\Sigma \frac{1}{d_n}$ be two Series of terms and $\Sigma 1/d_n$ be divergent also Suppose that

$$v_n = d_n \frac{u_n}{u_{n+1}} - d_{n+1},$$ then

i) If a fixed no. K can be formd so that after a certain stage, say for $n \geq m$, $v_n > K > 0$. the Series u_n is convergent

ii) If $v_n < 0$, for $n \geq m$, the Series Σu_n is divergent.

Proof :- Given,

$$v_n = d_n \frac{u_n}{u_{n+1}} - d_{n+1}$$

$$\Rightarrow v_n = \frac{d_n u_n - d_{n+1} u_{n+1}}{u_{n+1}}$$

① $v_n > K > 0$, $n \geq m$

$$\Rightarrow \frac{d_n u_n - d_{n+1} u_{n+1}}{u_{n+1}} > K, \quad n \geq m$$

$$\Rightarrow d_n u_n - d_{n+1} u_{n+1} > K u_{n+1}, \quad n \geq m$$

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putting n = $m, m+1, m+2, m+3, \ldots n-1$

$$d_m v_m - d_{m+1} v_{m+1} > K v_{m+1}$$

$$d_{m+1} v_{m+1} - d_{m+2} v_{m+2} > K v_{m+2}$$

$$d_{m+2} v_{m+2} - d_{m+3} v_{m+3} > K v_{m+3}$$

- - - - - - - - -

- - - - - - - - -

$$d_{n+1} v_{n-1} - d_n v_n > K v_n$$

on adding sidewise

$$d_m v_m - d_n v_n > K(v_{m+1} + v_{m+2} + v_{m+3} + \ldots + v_n)$$

$$\Rightarrow K(v_{m+1} + v_{m+2} + v_{m+3} + \ldots + v_n) < d_m v_m - d_n v_n$$

$$\Rightarrow K(v_{m+1} + v_{m+2} + v_{m+3} + \ldots + v_n) < d_m v_m$$

$$\Rightarrow v_{m+1} + v_{m+2} + v_{m+3} + \ldots + v_n < \frac{d_m v_m}{K}$$

$$\Rightarrow (v_1 + v_2 + v_3 + \ldots + v_m + v_{m+1} + v_{m+2} + v_{m+3} \ldots + v_n) - (v_1 + v_2 + v_3 + \ldots v_m) < \frac{d_m v_m}{K}$$

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Let S_n be the sum of n^{th} terms in Σu_n

$\therefore S_n - S_m < \frac{d_m v_m}{K}$

$\Rightarrow S_n < \left(S_m + \frac{d_m v_m}{K}\right)$

which is a fixed no.

$\therefore \Sigma u_n$ is convergent

(ii) $\because v_n < 0, \quad n \geq m$

$\therefore d_n \frac{v_n}{v_{n+1}} - d_{n+1} < 0, \quad n \geq m$

$\Rightarrow \frac{d_n v_n - d_{n+1} v_{n+1}}{v_{n+1}} < 0, \quad n \geq m$

$\Rightarrow d_n v_n - d_{n+1} v_{n+1} < 0, \quad n \geq m$

$\Rightarrow d_n v_n < d_{n+1} v_{n+1}, \quad n \geq m$

putting successively $n = m, m+1, m+2, \ldots\ldots n-1$

$d_m v_m < d_{m+1} v_{m+1} \ldots\ldots < d_n v_n$

$\Rightarrow d_m v_m < d_n v_n$

$\Rightarrow d_n v_n \not> d_m v_m$

$\Rightarrow v_n > d_m v_m \cdot \frac{1}{d_n}$

$\Rightarrow \sum_{n=1}^{\infty} v_n > d_m v_m \sum_{n=1}^{\infty} \frac{1}{d_n}$

Here $d_m v_m$ is a fixed no. & $\sum_{n=1}^{\infty} \frac{1}{d_n}$ is diverg.

Hence Σu_n is divergent

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Q. Examine the convergence of the following Series

$$\sum \frac{(n+1)(n+2)}{(n+3)(n+4)}$$

Soln:- Let u_n be the n^{th} term

$$\therefore u_n = \frac{(n+1)(n+2)}{(n+3)(n+4)}$$

$$\therefore u_{n+1} = \frac{(n+2)(n+3)}{(n+4)(n+5)}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{(n+1)(n+2)}{(n+3)(n+4)} \times \frac{(n+4)(n+5)}{(n+2)(n+3)}$$

$$\lim_{n\to\infty} \frac{n\left(1+\frac{1}{n}\right) \times n\left(1+\frac{5}{n}\right)}{n\left(1+\frac{3}{n}\right) \times n\left(1+\frac{3}{n}\right)}$$

$$= \frac{(1+0)\times(1+0)}{(1+0)\times(1+0)} = 1$$

$\therefore$ D'Alembert's ratio test fails

$\therefore$ We apply Rabee's test we get

$$\frac{u_n}{u_{n+1}} - 1 = \frac{(n+1)(n+2)}{(n+3)(n+4)} \times \frac{(n+4)(n+5)}{(n+2)(n+3)} - 1$$

$$= \frac{n^2+5n+n+5}{n^2+6n+9} - 1$$

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$$= \frac{n^2+6n+5 - n^2-6n-9}{n^2+6n+9}$$

$$= \frac{-4}{n^2+6n+9}$$

$$\Rightarrow n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{-4n}{n(n+6+9/n)}$$

$$\Rightarrow \lim_{n\to\infty} \left\{ n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} = \frac{-4}{(\infty+6+\infty)}$$

$$= 0 < 1$$

$\therefore \Sigma u_n$ is convergent. Proved

(82) Test the convergence of the series whose nth term u_n is given by

$$u_n = \frac{2^2.4^2.6^2 \cdots (2n-2)^2}{3.4.5 \cdots (2n-2)(2n-1)} x^{2n}.$$

Soln:-

Given $u_n = \frac{2^2.4^2.6^2 \cdots (2n-2)^2}{3.4.5}$

Given $u_n = \frac{2^2.4^2.6^2 \cdots (2n-2)^2}{3.4.5 \cdots (2n-2)(2n-1)} x^{2n}$

$\therefore u_{n+1} = \frac{2^2.4^2.6^2 \cdots (2n-2)^2 \{2(n+1)-2\}^2}{3.4.5 \cdots (2n-2)(2n-1)}$

$\{2(n+1)-2\}\{2(n+1)-1\}$

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$$= \frac{2^2 \cdot 4^2 \cdot 6^2 \cdots (2n-2)^2 (2n)^2}{3 \cdot 4 \cdot 5 \cdots (2n-2)(2n-1) \cdot 2n \cdot (2n+1)} \cdot x^{2n+2}$$

$$\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \lim_{n \to \infty} \frac{2^2 \cdot 4^2 \cdot 6^2 \cdots (2n-2)^2}{3 \cdot 4 \cdot 5 \cdots (2n-2)(2n+1)} \cdot x^{2n}$$

$$\times \frac{3 \cdot 4 \cdot 5 \cdots (2n-2)(2n-1) \cdot 2n (2n+1)}{2^2 4^2 6^2 \cdots (2n-2)^2 (2n)^2 \cdot x^{2n} \cdot x^2}$$

$$\lim_{n \to \infty} \frac{(4n^2 + 2n)}{4n^2} \cdot \frac{1}{x^2}$$

$$= \lim_{n \to \infty} \frac{x^2 \left(4 + \frac{2x}{n^2}\right)}{4n^2} \cdot \frac{1}{x^2}$$

$$= \frac{(4+0)}{4} \cdot \frac{1}{x^2} \Rightarrow \frac{1}{x^2}$$

By D'Alembert ratio test

if $x^2 < 1$ then $\frac{u_n}{u_{n+1}} > 1$

$\therefore \Sigma u_n$ is convergent

if $x^2 > 1$, then $\frac{u_n}{u_{n+1}} < 1$

$\therefore \Sigma u_n$ is divergent

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if $x^2 = 1$

then ratio test fails, we apply Rabee's test (putting $x^2 = 1$)

$$\frac{U_n}{U_{n+1}} = \frac{2n(2n+1)}{(2n)^2}$$

$$= \left(\frac{2n+1}{2n}\right)$$

$$\therefore \frac{U_n}{U_{n+1}} - 1 = \frac{2n+1}{2n} - 1$$

$$= \frac{2n}{2n} + \frac{1}{2n} - 1$$

$$= 1 + \frac{1}{2n} - 1$$

$$= \frac{1}{2n}$$

$$\Rightarrow n\left(\frac{U_n}{U_{n+1}} - 1\right) = n \times \frac{1}{2n}$$

$$= \frac{1}{2}$$

$$\Rightarrow \lim_{n \to \infty} n\left(\frac{U_n}{U_{n+1}} - 1\right) = \frac{1}{2} < 1$$

$\therefore$ by Robee's test

ΣU_n is divergent

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Q3) Test the convergence of the series

$$\frac{x}{2} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^3}{6} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{x^5}{10} + \cdots$$

Soln:- Let u_n be the nth term

$$u_n = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{(2n-1)}{2n}x^{2n-1}$$

$$\therefore u_{n+1} = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{(2n-1)}{2n}\times\frac{2(n+1)-1}{2(n+1)}x^{2(n+1)-1}$$

$$= \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{(2n-1)}{2n}\times\frac{(2n+1)}{(2n+2)}x^{2n+1}$$

$$\therefore \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{(2n-1)}{2n}x^{2n-1}}{\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{(2n-1)}{2n}\cdot\frac{(2n+1)}{(2n+2)}x^{2n+1}}$$

$$= \lim_{n\to\infty}\frac{x^{2n}\cdot x^{-1}}{\frac{(2n+1)}{(2n+2)}\cdot x^{2n}\cdot x}$$

$$= \lim_{n\to\infty}\frac{(2n+2)}{(2n+1)}\cdot\frac{1}{x^2}$$

$$= \lim_{n\to\infty}\frac{x(2+2/n)}{x(2+1/n)}\cdot\frac{1}{x^2}$$

$$= \frac{1}{x^2}$$

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By D'Alembert's ratio test

if

$x^2 < 1$

then $\frac{u_n}{u_{n+1}} > 1$

$\therefore \sum u_n$ is convergent

if $x^2 > 1$

then

$\frac{u_n}{u_{n+1}} < 1$

$\therefore \sum u_n$ is divergent

$\therefore \frac{u_n}{u_{n+1}} - 1$

$= \frac{1}{1} - 1$

$= 1$

Rabee's Test fails

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Q(4.) Test the convergence of the series

$$1 + a + \frac{a(a+1)}{1.2} + \frac{a(a+1)(a+2)}{1.2.3} + \ldots \text{ to } \infty$$

Soln :-

Let u_n be the nth term

$$u_n = \frac{a(a+1)(a+2)\ldots\{a+(n-1)\}}{1.2.3\ldots(n)a}$$

$$u_{n+1} = \frac{a(a+1)(a+2)\ldots\{a+(n+1)-1\}}{1.2.3\ldots(n+1)}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{\frac{a(a+1)(a+2)\ldots\{a+(n-1)\}}{1.2.3\ldots n}}{\frac{a(a+1)(a+2)\ldots\{a+(n+1)-1\}}{1.2.3\ldots(n+1)}}$$

$$= \frac{\{a+(n-1)\}}{n} \times \frac{(n+1)}{\{a+(n+1)-1\}}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{\{a+(1-\frac{1}{n})\}}{\infty} \times \frac{(1+\frac{1}{n})}{\{a+(1+\frac{1}{n})+1\}}$$

$$= \frac{a+(1-0)}{\infty} \times \frac{1+0}{a+1+0}$$

$$= \frac{a}{\infty} \times \frac{1}{a}$$

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$\log(1+x) = x + \frac{x^2}{2} + \frac{x^3}{3} \cdots$

V.V.I

Q5.) Test the convergence for positive real value of

$$x^2(\log e^2)^p + x^3(\log e^3)^p + x^4(\log e^4)^p + \cdots \infty$$

Soln:-

Let v_n be the n^{th} term

$$v_n = x^{n+1}\{\log e^{(n+1)}\}^p$$

$$v_{n+1} = x^{n+2}\{\log e^{(n+2)}\}^p$$

$$\therefore \lim_{n\to\infty} \frac{v_n}{v_{n+1}} = \frac{x^{n+1}\{\log e^{(n+1)}\}^p}{x^{n+2}\{\log e^{(n+2)}\}^p}$$

$$\lim_{n\to\infty} = \frac{x^n \cdot x}{x^n \cdot x^2}\left\{\frac{\log(n+1)}{\log(n+2)}\right\}^p$$

$$\Rightarrow \lim_{n\to\infty} = \frac{1}{x}\left[\frac{\log\{(n+2)-1\}}{\log(n+2)}\right]^p$$

$$\Rightarrow \lim_{n\to\infty} \frac{v_n}{v_{n+1}} = \frac{1}{x}\left[\frac{\log\{(n+2)\left(1-\frac{1}{n+2}\right)\}}{\log(n+2)}\right]^p$$

$$\lim_{n\to\infty} \frac{1}{x}\left[\frac{\log(n+2) + \log\left(1-\frac{1}{n+2}\right)}{\log(n+2)}\right]^p$$

$$\lim_{n\to\infty} \frac{1}{x}\left[1 + \frac{\log\left(1-\frac{1}{n+2}\right)}{\log(n+2)}\right]^p$$

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$$\lim_{n\to\infty} \frac{1}{n}\left[1 - \frac{1}{(n+2)\log(n+2)} - \frac{1}{2(n+2)^2\log(n+2)} - \cdots\right]$$

$$\lim_{n\to\infty} \frac{1}{n}\left[1 - \frac{p}{(n+2)\log(n+2)} - \frac{p}{2(n+2)^2\log(n+2)} - \cdots\right]$$

$$= \frac{1}{n}(1-0)$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{n}$$

By D'alembert's ratio test

if $n < 1$ then $\frac{u_n}{u_{n+1}} > 1$

$\therefore \sum u_n$ is convergent

if $n > 1$ then $\frac{u_n}{u_{n+1}} < 1$

$\therefore \sum u_n$ is divergent

if $n = 1$ then

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 1$$

this test is fails

$$\frac{u_n}{u_{n+1}} - 1 = \frac{1+p}{(n+2)\log(n+2)} - \frac{p}{2(n+2)^2\log(n+2)} - \cdots - 1$$

$$\Rightarrow n\left(\frac{u_n}{u_{n+1}} - 1\right) = -\frac{pn}{(n+2)\log(n+2)} - \frac{pn}{2(n+2)^2\log(n+2)}$$

$$\Rightarrow \lim_{n\to\infty}\left|n\left(\frac{u_n}{u_{n+1}} - 1\right)\right| = 0 < 1$$

By Rabee's test

$\therefore \sum u_n$ is divergent. ∎

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V.V.I

Q.) Test the convergence of the series whose nth term is given by

$$v_n = \left\{\frac{1.3.5.\ldots(2n-1)}{2.4.6.\ldots 2n}\right\}^2 \cdot \left(\frac{n}{n+1}\right)$$

Soln:

$$v_n = \left\{\frac{1.3.5.\ldots(2n-1)}{2.4.6.\ldots 2n}\right\}^2 \left(\frac{n}{n+1}\right)$$

$$v_{n+1} = \left\{\frac{1.3.5.\ldots(2n-1)\{2(n+1)-1\}}{2.4.6.\ldots 2n\{2(n+1)\}}\right\}^2 \cdot \frac{n+1}{n+2}$$

$$= \left\{\frac{1.3.5.\ldots(2n-1)(2n+1)}{2.4.6.\ldots 2n(2n+2)}\right\}^2 \cdot \frac{(n+1)}{(n+2)}$$

$$\therefore \lim_{n\to\infty} \frac{v_n}{v_{n+1}} = \lim_{n\to\infty} \left\{\frac{1.3.5.\ldots(2n-1)}{2.4.6.\ldots 2n}\right\}^2 \left(\frac{n}{n+1}\right) \left\{\frac{1.3.5.\ldots(2n-1)(2n+1)}{2.4.6.\ldots 2n(2n+2)}\right\}^2 \left(\frac{n+2}{n+1}\right)$$

$$\lim_{n\to\infty} \frac{n}{(n+1)} \cdot \frac{(2n+2)^2}{(2n+1)^2} \times \frac{(n+2)}{(n+1)}$$

$$\lim_{n\to\infty} \frac{n}{(n+1)^2} \cdot \frac{4(n+1)^2}{(2n+1)^2} \cdot (n+2)$$

$$\lim_{n\to\infty} \frac{4n^2+8n}{4n^2+4n+1}$$

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$$\lim_{n\to\infty} \frac{n^2\left(4+\frac{8}{n}\right)}{n^2\left(4+\frac{4}{n}+\frac{1}{n^2}\right)}$$

$$= \frac{4+0}{4+0+0} = \frac{4}{4} = 1$$

$\therefore$ D'Alembert's ratio test fails

$\therefore$ We apply Rabee's test

$$\frac{u_n}{u_{n+1}} - 1 = \frac{4n^2+8n}{4n^2+4n+1} - 1$$

$$= \frac{4n^2+8n-4n^2-4n-1}{(4n^2+4n+1)}$$

$$= \frac{4n-1}{(4n^2+4n+1)}$$

$$\Rightarrow n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{4n^2+n}{4n^2+4n+1}$$

$$\Rightarrow \lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} = \lim_{n\to\infty} \frac{n^2\left(4+\frac{1}{n}\right)}{n^2\left(4+\frac{4}{n}+\frac{1}{n^2}\right)}$$

$$= \frac{4-0}{4+0+0} = 1$$

$\therefore$ Rabee's test fails

then we apply Demorgen's & Bertrand's test

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{4n^2-n}{4n^2+4n+1}$$

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$$\Rightarrow n\left(\frac{u_n}{u_{n+1}}-1\right)-1 = \frac{4n^2-n}{4n^2+4n+1}-1$$

$$= \frac{4n^2-n-4n^2-4n-1}{4n^2+4n+1}$$

$$= \frac{-5n-1}{4n^2+4n+1}$$

$$\Rightarrow \left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n = \frac{n\left(-5-\frac{1}{n}\right)}{n^2\left(4+\frac{4}{n^2}+\frac{1}{n^2}\right)}\log n$$

$$\Rightarrow \lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n$$

$$= \lim_{n\to\infty}\left\{\frac{(-5-1/n)}{4+\frac{4}{n}+\frac{1}{n^2}}\right\}\lim_{n\to\infty}\frac{\log n}{n}$$

$$= \left(\frac{-5-0}{4+0+0}\right)\cdot 1$$

$$= -5/4 < 1$$

By De' morgon's test

Σu_n is divergent

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Q.) Define alternating Series :—

An infinite Series whose terms are alternately positive and negative, is called alternating Series.

An alternating Series is denoted by

$$\sum_{1}^{\infty} (-1)^{n-1} \cdot U_n = U_1 - U_2 + U_3 - U_4 - \cdots \text{ to } \infty$$

eg: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots$ to ∞

is alternating Series

Also,

$$1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots \text{ to } \infty$$

is not an alternating Series

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V.V.I.

Q// State and prove that Leibnitz's test

Statement :- An infinite series in which the terms are alternately positive and negative, is convergent each term is numerically less then the preceding term and the terms decrease infinitely

or,

Statement :- If for the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} v_n$

(i) $v_{n+1} < v_n$ for all n

(ii) $v_n \to 0$ as $n \to \infty$

then the series is convergent.

Proof :- Let the series is given by

$$v_1 - v_2 + v_3 - v_4 + v_5 - v_6 + \cdots + v_{2n-1} - v_{2n}$$

Let S_n be the sum of first $2n$ terms of given series

$$\therefore S_{2n} = v_1 - v_2 + v_3 - v_4 + v_5 - v_6 + \cdots + v_{2n-1} - v_{2n}$$

$$= (v_1 - v_2) + (v_3 - v_4) + (v_5 - v_6) + \cdots + (v_{2n-1} - v_{2n})$$

Here $v_{n+1} < v_n$

$\Rightarrow v_n > v_{n+1}$

$v_1 > v_2$, $v_3 > v_4$, $v_5 > v_6$ $\cdots$ $v_{2n-1} > v_{2n}$

$\Rightarrow v_1 - v_2$, $v_3 - v_4$, $v_5 - v_6$ $\cdots$ $v_{2n-1} - v_{2n}$ all are positive

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$\therefore S_{2n}$ is +ve and increase with n — (1)

Now $S_{2n} = U_1 - U_2 + U_3 - U_4 + U_5 - U_6 + \cdots + U_{2n-1} - U_{2n}$

$= U_1 - (U_2 - U_3 + U_4 - U_5 + U_6 + \cdots - U_{2n-1} + U_{2n})$

Here,

$U_2 - U_3$, $U_4 - U_5$ ... all are +ve

$\therefore S_{2n}$ is less then U_1

and $S_n > 0$

$\therefore 0 < S_{2n} \leq U_1$ — (2)

from (1) & (2)

the sequence $\{S_{2n}\}$ is monotonic increasing and bounded by 0 & 1.

$\therefore S_{2n}$ tends to a limit.

We have,

$S_{2n+1} > S_{2n} + U_{2n+1}$ $\left[\because S_n = S_{n-1} + U_n \right.$

$\left. \therefore U_n = S_n - S_{n-1}\right]$

$\Rightarrow \lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} U_{2n+1}$

$\Rightarrow \lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n+0}$ $\left[\because U_{n+0} \text{ as } n\to\infty \right.$

$\left. U_{2n+1} \to 0 \text{ as } n\to\infty\right]$

$\Rightarrow \lim_{n\to\infty} S_{2n}$

$\therefore S_{2n}$ tends to a finite limit

Hence, the series is convergent

Proved

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Q// Define an absolutely convergent Series:

An Series Σu_n whose terms are alternately +ve and -ve is said to be an obsolutely convergent Series if $\Sigma|u_n|$ is convergent. Series if $\Sigma|u_n|$ is convergent

e.g $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ to ∞

is an obsolutely Series

$\therefore 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$ to ∞ is a convergent Series

$u_n = \frac{1}{n}$

$\lim_{n\to\infty} u_n = \frac{1}{\infty}$

$= 0$

$\therefore \Sigma u_n$ is convergent.

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Q.) Define conditionally convergent Series or semi convergent Series -

A Series which terms alternately positive and negative is said to be semi convergent or non-obsolutely convergent or conditionally convergent Series if Σu_n is convergent but $\Sigma |u_n|$ is divergent

eg. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{\infty} = \Sigma u_n$

is a semi conductor Series

$$\sum_{n=1}^{\infty} \frac{1}{nP} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \frac{1}{\infty} = \Sigma |u_n|$$

Here, $P = 1$

$\Sigma |u_n| = \Sigma \frac{1}{nP}$ is divergent

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V.V.I

Theorem :— Prove that every absolutely convergent Series is convergent.

Proof :— Let U_n be an alternating Series which is absolutely convergent.

∴ By defination

$\Sigma|U_n|$ is convergent

Now,

$$U_n + |U_n| = \begin{cases} U_n + U_n, & n > 0 \\ U_n - U_n, & n < 0 \end{cases}$$

$$\Rightarrow U_n + |U_n| = \begin{cases} 2U_n, & n > 0 \\ 0, & n < 0 \end{cases} \quad \left[\because |n| = \begin{cases} n, & n > 0 \\ -n, & n < 0 \end{cases}\right.$$

Here $\Sigma U_n + |U_n| \geq 0$ and $\Sigma|U_n|$ is positive

Also, we have

$$\Sigma U_n + |U_n| \leq \Sigma 2|U_n|$$

and $\Sigma|U_n|$ is convergent

$\Rightarrow \Sigma 2|U_n|$ is convergent

∴ $\Sigma(U_n + |U_n|)$ is also convergent

$\Rightarrow \Sigma(U_n + |U_n| - |U_n|)$

$\Rightarrow \Sigma U_n$ is convergent

Proved

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Theorem - Prove that the terms of an absolutely convergent series can be arrangement without affecting the convergence and the sum of the series.

Proof :- Let Σu_n be a alternating series which is absolutely convergent

$\therefore$ by defination,

$\Sigma |u_n|$ is convergent.

Now $u_n + |u_n| = \begin{cases} u_n + u_n, & n \geq 0 \\ u_n - u_n, & n \neq 0 \end{cases}$

$u_n + |u_n| = \begin{cases} 2u_n, & n > 0 \\ 0, & n < 0 \end{cases}$

By the rearrangement of the term of Σu_n. we get a new series say Σv_n so that every v is u and every u is v

Here, $\Sigma v_n + |u_n| \geq 0$ and Σu_n is positive

Also we have

$\Sigma u_n + |u_n| \leq \Sigma 2|u_n|$

and $\Sigma |u_n|$ is convergent

$\Rightarrow \Sigma 2|u_n|$ is convergent

$\therefore \Sigma (u_n + |u_n|)$ is also convergent

Let $\Sigma |u_n| = S$

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and $\Sigma(v_{n+1} v_n)) = S$

$\therefore \Sigma(v_{n+1} v_n - v_n)) = S - S$

$\Sigma v_n = S - S$ — (i)

After re-arrangement all terms of v_n & v_n

$\Sigma|v_n| = S$

$\Sigma(v_n + |v_n|) = S$

$\therefore \Sigma(v_n + |v_n| - |v_n|) = S - S$

$\Rightarrow \Sigma v_n = S - S$ — (ii)

from eqn (i) & (ii)

$\Sigma v_n = \Sigma w_n$ Proved

Sequence

Q. Define Sequence or real sequence with ex.

Soln:-

A fun is whose domane is the set of natural no is known as a sequence. This is a real sequence u is a correspondence which associates each natural no. to uniquely determined real no. The image of the natural no. n is denoted by u_n or $u(n)$. Thus a sequence is denoted by $1, 2, 3, 4 \ldots$. Also a sequence is denoted by $\{u_n\}$ or $= \langle u_n \rangle$ or (u_n).

Ex - (1) $1, 3, 5 - - - 2n-1$, $n = 1, 2, 3 - -$

is a sequence

where, $u_n = 2n-1$

(2) $-1, +1, -1, +1 - - -$

is a sequence

where $u_n = (-1)^n$, $n = 1, 2, 3 -$

Q. Define a bounded above (upper)

A sequence $\{u_n\}$ is said to be bounded above if there exists a finite number $K \in R$, such that

$$u_n \leq K \quad \forall \; n \in N$$

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Q. Define bounded below, (lower)

A sequence $\{v_n\}$ is said to be bounded below if there exists a finite no. $K \in R$ such that

$$v_n \geq K \quad \forall \; n \in N$$

Q. Define a bounded sequence.

A sequence is said to be bounded if it is bounded above and below

i.e if K is upper bound & K is lower bound is sequence $\{v_n\}$ then $\{v_n\}$ is said to be bounded as $K \leq v_n \leq K \; \forall \; n \in N$

Q. Define least upper bound (l.u.b)

If $\{v_n\}$ be a sequence such that $v_n \leq M \; \forall \; n \in N$ and $v_n > m - \epsilon$ for atleast one value of $n \in N$, where ϵ is a positive no. however very small then M is called least upper bound.

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Q. Define greatest lower bound (g.l.b) :-
If $\{v_n\}$ be a sequence such that
$v_n \geq M \quad \forall\, n \in N$
and $v_n < m + \epsilon$ for atleast one value of $n \in N$, where ϵ is a positive no. however very small then M is called greatest lower bond

Ex: (i) $\{-v_n\} = \{-1, -2, -3, \ldots -n, \ldots\}$

is a sequence which is bounded above & unbounded below.

(ii) $\{v_n\} = \{1, 2, 3, \ldots n, \ldots\}$
is a sequence which bounded below and unbounded above

(iii) $\{v_n\} = \{2, \frac{3}{2}, \frac{4}{5}, \ldots \frac{n+1}{n}\}$

is a bounded sequence
upper bounded $= 2$
Lower bounded $= \lim_{n \to \infty} \left(\frac{n+1}{n}\right)$
$= \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)$
$= 1 + \frac{1}{\infty} = 1 + 0 = 1$

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Q.) Define monotonic increasing :—

A Sequence $\{v_n\}$ is said to be monotonic increasing Sequence if,

$v_{n+1} \geq v_n \quad \forall n \in I$

i.e $\{v_n\}$ is a monotonic increasing Sequence if $v_1 \leq v_2 \leq v_3 < v_4 \leq \cdots \leq v_n \leq \cdots$

Q.) Define Strictly monotonic increasing :—

A Sequence $\{v_n\}$ is said to be Strictly monotonic increasing Sequence if $a_{n+1} > v_n$

i.e $\{v_n\}$ is a monotonic increasing Sequence if $v_1 < v_2 < v_3 \cdots < v_n < \cdots$

Q. Define monotonic decreasing :—

A Sequence $\{v_n\}$ is said to be monotonic decreasing Sequence if

$v_{n+1} \leq v_n \quad \forall \; n \in N$

i.e $\{v_n\}$ is a monotonic decreasing Sequence

if $v_1 \geq v_2 \geq v_3 \geq v_4 \cdots v_n \geq \cdots$

Q. Define limit of Sequence :—

A Sequence is said to be have a limit l if for positive integer m,

Such that

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$|v_n - l| < \epsilon$ as $n \geq m$

$\Rightarrow l - \epsilon < v_n < l + \epsilon$ as $n \geq m$

or, A sequence (v_n) is said to have a limit l if

$$\lim_{n \to \infty} v_n = l.$$

Q. Define limit point of sequence:-

A real no. l each said to be the limit point of a sequence if every ϵ neighbourhood of l contains an infinite no. of terms of the sequence.

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V.V.I.

Bolzano-Weierstrass's theorem

Q. Theorem :- state and prove the Bolzano Weierstrass theorem.

Statement :-

Every bounded sequence has a limit point.

Proof :- let v_n is a bounded sequence and u be the set of terms of the sequence.

i.e $u = \{v_1, v_2 \dots v_n \dots\}$

if $\{v_n\}$ has infinity many distinct point then the result follows from Bolzano Weierstrass an limit point

if Σv_n have finite no. of distinct point and rest all are equal. then let $v_n = v_{n+1}$ from $n \geq n_1$

$\therefore u = \{u_1, v_2, v_3 \dots v_n\}$

every neighbourhood of v_m contains all element of V except perhaps $\{v_1, v_2, v_3 \dots v_n \dots\}$

for $u_m - \epsilon < x < v_m + \epsilon$

for $x = v_n$ where $n \geq m$

hence v_m is a limit point if the Sequence $\{v_n\}$ proved

OR

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$\frac{1}{2n}$ $(2)^{1/n}$

8. Define convergent sequence:-

A sequence $\{v_n\}$ is said to be convergent if $v_n \to l$, a finite limit as $n \to \infty$

OR

A sequence $\{v_n\}$ is said to be convergent to l if for every $\epsilon > 0$ there exist a +ve integer m. such that

$$|v_n - l| < \epsilon \quad \text{as} \quad n \geq m$$

or, $$l - \epsilon < v_n < l + \epsilon \quad \text{as} \quad n > m$$

Ex:- $\frac{1}{2}, \frac{3}{4}, \frac{5}{6} \dots \frac{2n-1}{2n} \dots$

Here, $v_n = \frac{2n-1}{2n}$

$$|v_n - 1| = \left| \frac{2n-1}{2n} - 1 \right|$$

$$= \left| 1 - \frac{1}{2n} - 1 \right| = 2^{1/n}$$

$\Rightarrow |v_n - 1| < \epsilon \quad , \epsilon > \frac{1}{2^n}$

$\Rightarrow |v_n - 1| < \epsilon \quad , n > \frac{1}{2\epsilon}$

$\Rightarrow |v_n - 1| < \epsilon \quad \text{as} \quad n > m \quad \left(\because m = \frac{1}{2\epsilon} \right.$

$\therefore \{v_n\}$ is convergente.

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Q. Theorem :- state and prove that cauchy's general principle of convergence of a sequence.

Statement :- The necessary and sufficient condition for the convergence of the sequence $\{a_n\}$ is that corresponding to an arbitrary positive number ϵ, however small there exists a positive integer v such that $|a_{n+p} - a_n| < \epsilon$,

for all values of $n \geq v$ and for all +ve integral value of p.

Proof :- necessary condition :-

Let $\{a_n\}$ be a convergent sequence which convergent to a finite limit l (say) Then for every $\epsilon > 0$ there exists a +ve integer v such that.

$$|a_n - l| < \frac{\epsilon}{2} \quad \text{al} \quad n \geq v \quad \text{—} (1)$$

Also, for all $n \geq v$ & if p is +ve integer then,

$$|a_{n+p} - l| < \frac{\epsilon}{2} \quad \text{—} (2)$$

Now, $a_{n+p} - a_n = a_{n+p} - l - a_n + l$

$$= (a_{n+p} - l) - (a_n - l)$$

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$\Rightarrow |a_{n+p} - a_n| < |a_{n+p} - l| + |a_n - l|$

$= \frac{\epsilon}{2} + \frac{\epsilon}{2}$ (from eq' ① & ②)

$= \epsilon$

$= |a_{n+p} - a_n| < \epsilon$ $\forall$ $n \geq \nu$.

Sufficient condition:-

we have

$|a_{n+p} - a_n| < \epsilon$ for $n \geq \nu$ & p is +ve int.

$\Rightarrow a_n - \epsilon < a_{n+p} < a_n + \epsilon$, $n \geq \nu$ — ①

& p is a +ve integer

Let the sequence $\{a_{n+p}\}$, $p = 1, 2, 3$ — is bdd

Let the greatest lower bond and least upper bound be m & M respectively, such that

$m > a_n - \epsilon$

& $M < a_n + \epsilon$

$\therefore M - m = (a_n + \epsilon) - (a_n - \epsilon)$

$= 2\epsilon$

Here ϵ is an arbitrary & however very small

$\therefore M - m = 0$

$\Rightarrow M = m$

$\therefore M - \epsilon < a_{n+p} < M + \epsilon$

$\Rightarrow |a_{n+p} - M| < \epsilon$, $n \geq \nu$

$\Rightarrow \{a_{n+p}\}$ convergent & $\to M$

Hence $\{a_n\}$ is convergent

proved

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Q Define cauchy Sequence OR fundamental Sequence :-

A Sequence $\{a_n\}$ is said to be a cauchy Sequence if for every $\epsilon > 0$ there exists a +ve number is such that

$$n, m \geq \nu$$
$$\Rightarrow |a_n - a_m| < \epsilon$$

Imp. (5)

Q. Theorem :- Prove that every cauchy Sequence is convergent in R.

Proof :- Let $\{a_n\}$ be a cauchy Sequence for every $\epsilon > 0$ there exists a +ve integer is such that

$$n, m \geq \nu$$
$$\Rightarrow |a_n - a_m| < \epsilon \quad \text{— (i)}$$

putting $m = n+p$ in (i) we get

$$|a_n - a_{n+p}| < \epsilon$$

$$\Rightarrow |a_{n+p} - a_n| < \epsilon \text{ as } n \geq \nu$$

& p is a +ve integer

by the cauchy's genral principle of convergence the

Sequence $\{a_n\}$ is convergent.

Proved

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(Cauchy) A sequence a_n is said to be Cauchy sequence

(4)

8. Theorem :- Prove that every convergent sequence is cauchy in R.

proof :- let $\langle a_n \rangle$ be a convergent sequence which convergent to a limit l.

for every $\epsilon > 0$ there exist $\mu > 0$ such that

$n \geq \mu$

$\Rightarrow |a_n - l| < \frac{\epsilon}{2}$ — (1)

& $\forall$ $m \geq \mu$

$\Rightarrow |a_m - l| < \frac{\epsilon}{2}$ — (2)

Now $a_n - a_m = a_n - l - a_m + l$

$= (a_n - l) - (a_m - l)$

$\therefore |a_n - a_m| < |(a_n - l) - (a_m - l)|$

$= \frac{\epsilon}{2} + \frac{\epsilon}{2}$ [from (1) & (2)]

$= \epsilon$

$\therefore |a_n - a_m| < \epsilon$

$\therefore$ for $\epsilon > 0$ there exist n, m such that

$n, m \geq \mu$

$\Rightarrow |a_n - a_m| < \epsilon$

$\therefore$ By defination

$\{a_n\}$ is cachy sequence

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V.V.I

V.V.I

Q Theorem :- prove that every cauchy sequence is bounded in R.

Soln :-

Proof :- Let $\{a_n\}$ be a cauchy sequence then we have to show that $\{a_n\}$ is bounded in R.

By defination of cauchy sequence we have for every $\epsilon > 0$ there exists a +ve no. ν such that

$m, n \geq \nu$

$\Rightarrow |a_n - a_m| < \epsilon$

for a particular value

We take a no. K for m greater the ν, we get

$|a_n - a_K| < \epsilon$ for $n \geq \nu$

$\Rightarrow a_K - \epsilon < a_n < a_K + \epsilon$ for $n \geq \nu$

Let $\lambda = \min [a_1, a_2, a_3 \cdots a_{\nu-1}, a_K - \epsilon]$

$\mu = \max \{a_1, a_2, a_3 \cdots a_{\nu-1}, a_K + \epsilon\}$

from (1)

$\lambda < a_n < \mu$ for $n \geq \nu$

$\Rightarrow \{a_n\}$ bounded

Proved

V.V.I

Q. Theorem: Prove that a convergent sequence determine its limit uniquely.

OR

Prove that every convergent sequence has some limit

proof:- Let $\{a_n\}$ be a convergent sequence if possible then let l & l' be any two limits in $\{a_n\}$ then we have to show that

$$l = l'$$

∴ By defination of convergent sequence for every $\epsilon > 0$ there exists a +ve no. such that

$$|a_n - l| < \frac{\epsilon}{2} \quad \text{for } n \geq m \quad \text{---} (1)$$

& $$|a_n - l'| < \frac{\epsilon}{2} \quad \text{for } n \geq m \quad \text{---} (2)$$

Now $$l - l' = l - a_n + a_n - l'$$

$$= -(a_n - l) + (a_n - l')$$

$$\Rightarrow |l - l'| \leq |a_n - l| + |a_n - l'|$$

$$= \frac{\epsilon}{2} + \frac{\epsilon}{2} \quad \text{from (1) \& (2)}$$

$$\therefore |l - l'| < \epsilon \quad \text{for } n \geq m$$

but ϵ is an arbitrary however very small

$$l - l' = 0$$

$$\Rightarrow \boxed{l = l'}$$ Proved

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Theorem :- Prove that every convergent sequence is bounded.

Proof :- Let $\{a_n\}$ be a convergent sequence which convergent to a finite limit l as $n \to \infty$

for every $\epsilon > 0$ there exists a +ve no. in such that

$$|a_n - l| < \epsilon \quad \text{as } n \geq m$$

$$\Rightarrow l - \epsilon < a_n < l + \epsilon \quad \text{as } n \geq m$$

We take,

a finite no. of elements a_1, $a_1, a_2, a_3 - \cdots a_{n-1}$

Let G & g be the greatest and least element among then.

We take

$\therefore M$ = greatest of G an $l + \epsilon$

$= \max\{a_1, a_2, a_3 - a_{n-1}, l+\epsilon\}$

λ = least of g and $l - \epsilon$

$= \min\{a_1, a_2, a_3 - \cdots a_{n-1}, l-\epsilon\}$

$\therefore \lambda \leq a_n \leq M$ for $n \geq m$

$\Rightarrow \{a_n\}$ is bounded. proved

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Theorem :- prove that every monotonic increasing sequence tends to its least upper bound.

proof :- Let $\{a_n\}$ be a monotonic increasing sequence.

Also M be the least upper bound in $\{a_n\}$

$\therefore a_n \leq M$ for all $n \in N$ — (1)

& $a_n + \epsilon > M$ for atleast one $n \in N$ — (2)

taking $n = m$ in (2) we get

$a_m + \epsilon > M$

$\Rightarrow a_m > M - \epsilon$ — (3)

Now,

$\because \{a_n\}$ a monotonic increasing sequence

$\therefore a_n \geq a_m$, $n \geq m$

$\therefore a_n > m - \epsilon$ (from 3) — (4)

from (1)

$a_n \leq M$

$\Rightarrow a_n < M + \epsilon$ — (5)

from (4) & (5) we get

$m - \epsilon < a_n < m + \epsilon$

$\Rightarrow |a_n - m| < \epsilon$ $\forall$ $n \geq m$

$\Rightarrow a_n \to M$ $\forall$ $n \geq m$ proved

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Theorem :- Prove that every monotonic decreasing sequence tends to greatest lowest bound.

Proof :- Let $\{a_n\}$ be a monotonic decreasing sequence.

Also K be the greatest lower bound in $\{a_n\}$

$\therefore a_n \geq K$ for all $n \in N$ — (1)

& $a_n - \epsilon < K$ for atleast one $n \in N$ — (2)

taking $n = m$ in (2)

$a_m - \epsilon < K$

$\Rightarrow a_m < K + \epsilon$ — (3)

Now $\because \{a_n\}$ is a monotonic decreasing sequence

$\therefore a_n \leq a_m, \quad n \geq m$

$\therefore a_n < K + \epsilon$ (from 3) — (4)

from (1) $a_n \geq K$

$\Rightarrow a_n > K - \epsilon$ — (5)

from (4) & (5)

$K - \epsilon < a_n < K + \epsilon$

$\Rightarrow |a_n - K| < \epsilon$ as $n \geq m$

$\Rightarrow a_n \to K$ as $n \geq m$

Proved

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(8) If $u_n = \frac{2n+1}{3n+5}$ show that $\lim_{n\to\infty} u_n = 2/3$

Soln:- by defination of limit

$$u_n = \frac{2n+1}{3n+5}$$

$$\therefore |u_n - 2/3| < \epsilon$$

$$\Rightarrow \left| \frac{2n+1}{3n+5} - \frac{2}{3} \right| < \epsilon$$

$$\Rightarrow \left| \frac{6n+3-6n-10}{9n+15} \right| < \epsilon$$

$$\Rightarrow \left| \frac{-7}{9n+15} \right| < \epsilon$$

$$\Rightarrow \frac{7}{9n+15} < \frac{\epsilon}{1}$$

$$\Rightarrow \frac{9n+15}{7} > \frac{1}{\epsilon}$$

$$9n + 15 > \frac{7}{\epsilon}$$

$$\Rightarrow 9n > \frac{7}{\epsilon} - 15$$

$$\Rightarrow n > \frac{7}{9\epsilon} - \frac{15}{9}$$

Here, m is a +ve integer equal & greater than $\frac{7}{9\epsilon} - \frac{15}{9}$

$\therefore |u_n - \frac{2}{3}| < \epsilon$ as $n \geq m$

Hence $\lim_{n\to\infty} u_n = 2/3$ proved.

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अभिजन की धरती के लंबस आर, केपर्ड भी कहते हैं
ATM - जान शेपर्ड - (1965)

(8) Prove that the sequre $\{u_n\}$ converge to a limit lying b/w 2 & 3 where

$$u_n = \left(1 + \frac{1}{n}\right)^n$$

Soln:- $u_n = \left(1 + \frac{1}{n}\right)^n$

$$= 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{\lfloor 2} \cdot \frac{1}{n^2} + \frac{n(n-1)(n-2)}{\lfloor 3} \cdot \frac{1}{n^3}$$

$$+ \cdots + \frac{n(n-1)(n-2)\cdots 1}{\lfloor n} \cdot \frac{1}{n^n} + \cdots$$

(Using binomial th)

$$\Rightarrow u_n = 2 + \frac{n(n-1)}{\lfloor 2} \frac{1}{n^2} + \frac{n(n-1)(n-2)}{\lfloor 3} \cdot \frac{1}{n^3}$$

$$+ \cdots + \frac{n(n-1)(n-2)\cdots 1}{\lfloor n} \cdot \frac{1}{n^n} + \cdots$$

$$\Rightarrow u_n = 2 \quad \text{---} (1)$$

Also,

$$u_n < 1 + 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \cdots + \frac{1}{\lfloor n}$$

$$\Rightarrow u_n < 1 + \left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}\right)$$

$$\Rightarrow u_n < 1 + \left(\frac{1 - \frac{1}{2^n}}{1 - 1/2}\right) \quad \{\text{Series in G.P } 1 - R_n\}$$

$$\Rightarrow u_n < 1 + \left(\frac{1 - 1/2^n}{1/2}\right)$$

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$\Rightarrow u_n < 2 + 2\left(1 - \frac{1}{2^n}\right)$

$\Rightarrow u_n < 1 + 2 \quad [n \to \infty]$

$\Rightarrow u_n < 3 \quad \text{---} (2)$

equal (1) & (11)

$2 < u_n < 3$

$\Rightarrow u_n$ lies b/w 2 & 3 proved

V.V.I.

Q. Prove that the sequence $\{a_n\}$ defined by the relation

$a_n = 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \cdots + \frac{1}{\lfloor n-1}$

is convergent.

Soln:- Given,

$a_n = 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \cdots + \frac{1}{\lfloor n-1}$

$a_{n+1} = 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \cdots + \frac{1}{\lfloor n-1} + \frac{1}{\lfloor n}$

Now,

$a_{n+1} - a_n = 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \cdots + \frac{1}{\lfloor n-1} + \frac{1}{\lfloor n}$

$- 1 - \frac{1}{\lfloor 1} - \frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} - \cdots - \frac{1}{\lfloor n-1}$

$\Rightarrow a_{n+1} - a_n = \frac{1}{\lfloor n} > 0 \quad \text{---} (1)$

$a_{n+1} - a_n > 0$

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$\Rightarrow a_{n+1} > a_n$

$\therefore$ The sequence $\{a_n\}$ is monotonic increasing sequence.

If $n=0$ in (1) then

$$a_1 - a_0 = \frac{1}{\lfloor 0}$$

$$\Rightarrow a_1 = a_0 + 1 \quad (\because a_0 = 1)$$
$$= 1+1$$
$$= 2$$

Now,

$$a_n = 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \cdots + \frac{1}{\lfloor n-1}$$

$$< 1 + \left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}\right)$$

$$< 1 + 1 \cdot \frac{\left(1 - \frac{1}{2^{n-1}}\right)}{\left(1 - \frac{1}{2}\right)}$$ (Series in GP with Comon ratio 1/2)

$$< 1 + 2\left(1 - \frac{1}{2^{n-1}}\right)$$

$$< 1 + 2$$

$$\Rightarrow a_n < 3 \quad \text{---(2)}$$

$\therefore 0 < a_n < 3$ (from (1))

$\therefore a_n$ is bounded

Thus, $\{a_n\}$ is monotonic increasing and bounded sequence

Hence a_n is convergent.

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Q. Prove that the sequence $\{a_n\}$ is defined by the relation

$$a_n = 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \cdots + \frac{1}{\lfloor n}$$

is convergent.

Soln:-

Given

$$a_n = 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \cdots + \frac{1}{\lfloor n}$$

$$a_{n+1} = 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \cdots + \frac{1}{\lfloor n} + \frac{1}{\lfloor n+1}$$

Now,

$$a_{n+1} - a_n = 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \cdots + \frac{1}{\lfloor n} + \frac{1}{\lfloor n+1}$$

$$- 1 - \frac{1}{\lfloor 1} - \frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} - \cdots - \frac{1}{\lfloor n}$$

$$a_{n+1} - a_n = \frac{1}{\lfloor n+1} > 0 \quad \text{— (1)}$$

$$\Rightarrow a_{n+1} - a_n > 0$$

$$\Rightarrow a_{n+1} > a_n$$

$\therefore$ The sequence $\{a_n\}$ is monotonic increasing sequence

If $n = 0$ in (1) then

$$a_1 - a_0 = \frac{1}{1}$$

$$\Rightarrow a_1 = a_0 + 1$$
$$= 1 + 1$$
$$= 2$$

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Now, $a_n = 1 + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \cdots + \frac{1}{15}$

$= 11 + \left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}\right)$

$= 11 + 1\left(\frac{1 - \frac{1}{2^n}}{1 - 1/2}\right)$

(Series in G.P with common ratio $1/2$)

$11 + 1\left(\frac{1 - 1/2^n}{1/2}\right)$

$11 + 2\left(1 - \frac{1}{2^n}\right)$

$\Rightarrow v_n < 3$ — (2)

$\therefore 0 < a_n < 3$ (from (2))

$\therefore a_n$ is bounded

Thus, $\{a_n\}$ is monotonic increasing and bounded.

Hence $\{a_n\}$ is convergent

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Q Show that $\lim_{n\to\infty} n^{1/n} = 1$

Soln:- Let n^{th} term is denoted by v_n.

$\therefore v_n = n^{1/n} \quad \forall\ n \in N$

We know that

$$\left(1+\frac{1}{n}\right)^n < 3 \quad \text{for all } n$$

$$\Rightarrow \left(1+\frac{1}{n}\right)^n < 3 \leq n \quad \forall\ n.$$

$$\Rightarrow n > \left(1+\frac{1}{n}\right)^n$$

$$\Rightarrow n > \frac{(n+1)^n}{n^n}, \quad \forall\ n$$

$$\Rightarrow n^1 \cdot n^n > (n+1)^n$$

$$\Rightarrow n^{(1+n)} > (n+1)^n$$

$$\Rightarrow n^{1/n} > (n+1)^{1/n+1}$$

$$\Rightarrow v_n > v_{n+1}, \quad \forall\ n$$

$\therefore \{v_n\}$ is monotonic decreasing

Also, $v_n = n^{1/n}$

$v_1 = 1$

$v_2 = 2^{1/2} > 1$

$v_3 = 3^{1/3} > 1$

Thus 1 is the greatest lower bound $\forall\ n$,

$$\lim_{n\to\infty} v_n = 1$$

$$\Rightarrow \lim_{n\to\infty} n^{1/n} = 1$$

proved

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(8) Prove that the sequence $\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}$ convergence to 2.

Soln:

Let a_n be the nth term of given sequence

$\therefore a_n = \sqrt{2a_{n-1}}$

Squaring both sides, we get

$a_n^2 = 2a_{n-1}$ — (1)

replacing n by n-1 we get

$a^2_{n-1} = 2a_{n-2}$ — (2)

from (1)

$a_n^2 = 2a_{n-1}$

$\Rightarrow \frac{a_n}{a_{n-1}} = \frac{2}{a_n}$

If $a_n > a_{n-1}$

then $2 > a_n$

$\Rightarrow a_n < 2$

Subtracting (1) & (2)

$a_n^2 - a^2_{n-1} = 2a_{n-1} - 2a_{n-2}$

$= 2(a_{n-1} - a_{n-2})$

So that $a_n >$ or $< a_{n-1}$

a/c

$\therefore \{a_n\}$ is monotonic increasing & bounded above by 2.

So the sequence $\{a_n\}$ converges to a limit l (say)

$\therefore \lim_{n\to\infty} a_n = l$ & $\lim_{n\to\infty} a_{n-1} = l$

from (1)

$l = 2l$

Hence the sequence $\{a_n\}$ converges to 2.

Proved

(8) Prove that the sequence $\sqrt{5}, \sqrt{(5\sqrt{5})}, \sqrt{(5\sqrt{(5\sqrt{5})})}$ convergent to 5.

Soln:-

Let a_n be the nth term of given Sequence

$\therefore a_n = \sqrt{5a_{n-1}}$

Squaring both sides we get

$a_n^2 = 5a_{n-1}$ — (1)

replacing n by n-1 we get

$a_{n-1}^2 = 5a_{n-2}$ — (1)

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Subtracting (1) & (2)

$$a_n^2 - a_{n-1}^2 = 5a_{n-1} - 5a_{n-2}$$
$$= 5(a_{n-1} - a_{n-2})$$

so that

$$a_n > \text{or} < a_{n-1}$$

& $a_{n-1} > \text{or} < a_{n-2}$

from (1)

$$a_n^2 = 5a_{n-1}$$

$$\Rightarrow \frac{a_n}{a_{n-1}} = \frac{5}{a_n}$$

if $a_n > a_{n-1}$

Then $5 > a_n$

$\therefore \{a_n\}$ is monotonic increasing & bounded above by 5.

So the sequence $\{a_n\}$ converges to a limit l

$$\therefore \lim_{n\to\infty} a_n = l \quad \& \quad \lim_{n\to\infty} a_{n-1} = l$$

from (1)

$$l^2 = 5l$$

$$l = 5$$

Hence, the sequence $\{a_n\}$ converge to 5.

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(8) Prove that the sequence $\sqrt{7}, \sqrt{(7\sqrt{7})}, \sqrt{(7(7\sqrt{7}))}$, convergent to 7.

Soln:-

Let a_n be the nth term of the given sequence

$\therefore a_n = \sqrt{7a_{n-1}}$

Squaring both sides, we get

$a_n^2 = 7a_{n-1}$ ——— (1)

replacing n by n-1 we get

$a_{n-1}^2 = 7a_{n-2}$ ——— (2)

Substracting (1) & (2)

$a_n^2 - a^2_{n-1} = 7a_{n-1} - 7a_{n-2}$

$= 7(a_{n-1} - a_{n-2})$

So that

$a_n > \text{or} < a_{n-1}$

i/e $a_{n-1} > \text{or} < a_{n-2}$

from (1)

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$$a_n^2 = 7a_{n-1}$$

$$\Rightarrow \frac{a_n}{a_{n-1}} = \frac{7}{a_n}$$

if $a_n > a_{n-1}$

Then $7 > a_n$

$\therefore \{a_n\}$ is monotonic increasing & bounded above by 7.

So the sequence $\{a_n\}$ convergence to a limit l

$$\lim_{n\to\infty} a_n = l \quad \& \quad \lim_{n\to\infty} a_{n-1} = l$$

from (1)

$$l^2 = 7l$$

$$\therefore l = 7$$

Hence the Sequence $\{a_n\}$ conroge to 7.

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8 Prove that the sequence $\sqrt{7}, \sqrt{7+\sqrt{7}}, \sqrt{7+\sqrt{7+\sqrt{7}}}$ — tends to the positive root of $x^2 - x - 7 = 0$ as limit.

Soln :-

Let a_n be the nth term of given sequence

$$a_n = \sqrt{7 + a_{n-1}}$$

Squaring both sides, we get

$$a_n^2 = 7 + a_{n-1} \quad \text{— (1)}$$

replacing n by n-1 we get

$$a_{n-1}^2 = 7 + a_{n-2} \quad \text{— (2)}$$

Subtracting (1) & (2) we get

$$a_n^2 - a_{n-1}^2 = (7 + a_{n-1}) - (7 + a_{n-2})$$

$$= 7 + a_{n-1} - 7 - a_{n-2}$$

$$= a_{n-1} - a_{n-2}$$

Here $a_n > \text{or} < a_{n-1}$

according as $a_{n-1} > \text{or} < a_{n-2}$

Here, $a_1 = \sqrt{7}$, $a_2 = \sqrt{7+\sqrt{7}}$

$\therefore a_2 > a_1$

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$\therefore \{a_n\}$ is monotonic increasing

We shall show that $\{a_n\}$ is bounded above

$\therefore \sqrt{7} < \sqrt{7+\sqrt{7}} < \sqrt{7}$

$\sqrt{7+\sqrt{7}} < \sqrt{7+7} = \sqrt{14} < 7$

$\therefore a_n \leq 7$

$\therefore \{a_n\}$ is bounded above by 7

$\therefore \{a_n\}$ is convergent

Which converges to a finite limit l (say)

i.e $a_n \to l$ & $a_{n-1} \to l$

as $n \to \infty$

from (1)

$l^2 = 7 + l$

$\Rightarrow l^2 - l - 7 = 0$

replacing l by x we get

$\underline{x^2 - x - 7 = 0}$

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(8) Define null sequence

→ A sequence $\{a_n\}$ is said to be a null sequence if $a_n \to 0$ as $n \to \infty$

eg:- Show that the sequence $\{a_n\} = \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots \frac{1}{2^n}, \ldots$ is a null sequence

Here $a_n = \frac{1}{2^n}$

taking limit on both sides a

$$\lim_{n \to \infty} a_n$$

$$\lim_{n \to \infty} \frac{1}{2^n} = 0$$

$\Rightarrow a_n \to 0$ as $n \to \infty$

Hence, sequence $\{a_n\}$ is a null seq

Q Theorem: State and prove the sequence prin
Statement :-

If $\{a_n\}, \{b_n\}, \{c_n\}$ are sequence
Such that $a_n \leq b_n \leq c_n$ for all
values of n and

$$\lim_{n\to\infty} a_n = l \quad \lim_{n\to\infty} c_n = l$$

then $\lim_{n\to\infty} b_n = l$

Proof: Let $\{a_n\}, \{b_n\}, \{c_n\}$ be any three sequence such that

$$a_n \leq b_n \leq c_n \quad \forall n$$

& $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = l$

then we have to prove that

$$\lim_{n\to\infty} b_n = l$$

$\therefore$ for every $\epsilon > 0$

$\because \lim_{n\to\infty} a_n = l$

$\therefore l - \epsilon < a_n < l + \epsilon$ for $n \geq n_1$.

& $\lim_{n\to\infty} c_n = l$

$\therefore l - \epsilon < c_n < l + \epsilon$, for $n \geq n_2$

Now $\because a_n < b_n$

$\therefore l - \epsilon < b_n$ for $n \geq n_1$.

& $b_n < c_n$

$\because b_n < l + \epsilon$ for $n \geq n_2$

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thus we get

$l - \epsilon < b_n < l + \epsilon$ as $n \geq n_0$

$\therefore n_0$ is greatar than n_1 & n_2 (max)

$\Rightarrow |b_n - l| < \epsilon$ as $n \geq n_0 = \lim_{n \to \infty} b_n = l$

Proved.

① Define a set bounded above.

⇒ A subset S of R is said to be bounded above if there exists a real number K such that

$$x \leq K \quad \forall \quad x \in S$$

Here, the real no. K is called an upper bound of S.

ex: (I) The sets $(1,2)$ $[1,2]$, $[1,2)$, $(1,2]$ are bounded above

(II) The set of -ve integers is bounded above by 0.

Definition :- Define a set bounded below.

A subset S of R is said to be bounded below if there exists a real no. K such that

$$x \geq K \quad \forall \quad x \in S$$

Here, The real no. K is called a lower bound of S.

ex. The sets $(1,2)$, $[1,2]$, $[1,2)$, $(1,2$ are bounded below

and $\forall$ N are bounded below.

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Q. Define bounded set.

A subset S of R is said to be a bounded set if it is bounded below as well as above that is there exist two real no. k & K

$$k \leq x \leq K \quad \forall\ x \in S$$

eg. (i) The sets $(1,2)$, $[1,2]$, $[1,2)$, $(1,2]$ are bounded.

(ii) every finite set is bounded

(iii) The set $\{1/2, 1/3, 1/4 \ldots\}$ is bounded

Q.) Define unbounded set.

→ A subset S of R which is not bounded that is which, is neither bounded above nor bounded below, is called an unbounded set

eg. (i) N, Z, Q, R are unbounded sets

(ii) Set of all prime no. is an unbounded set.

Q. Define supremum or least upper bound or l.u.b

→ Let S be a non empty subset of R then a real no. M is said to be a least upper bound of S if

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eg. (i) u is an upper bound of S.
i.e $x \leq u \quad \forall x \in S$

(ii) if u is another upper bound of S.
then $u < u'$ i.e no. real no. less then u
can be an upper bound of S.

⑤ Define greatest lower bound or g.l.b or infimum.

Sol^n:- Let S be a non-empty subset of R then a real no. l is called a greatest lower bound of S if

eg. (i) l is a lower bound of S i.e $l \leq x$ $\forall x \in S$

(ii) if l' is another lower bound of S. then $l' \leq l$
i.e No real no. $r \geq l$ can be a lower bound of S.

⑥ Write the completness Property or Axiom of R.

Sol^n:- The Set of all upper bounds of a non-empty and bounded above real no. has the smallest element. we can say that every non-empty subset of reals which is bounded above has the least upper bound.

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of the set of all lower bounds of a non-empty set of reals which is bounded below has the greatest element then that element is called the greatest lower bound of S g.l.b or infimum of S.

(8) Write the Archimedean property.
⇒ Let x & y be any two real numbers which x > 0 then there exists a natural no. n such that nx > y

(b) Write the properties of real nos.
⇒ The properties of real nos. are given as

(i) Archimedean property :- Let x & y be any two real nos. with x > 0 then there exists a natural no. n such that nx > y

(ii) Given any real no. x there exists a natural no. n such that n > x

(iii) for each +ve real no. x. There exists n belong N such that x > 1/n

(iv) Given any x belong to R there exists integer p & q such that q > 1/p

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(v) Given any x belong to R there exists any ~~die infinitely many~~ rational nos. unique integer m, $m+1 \geq x \geq m$

(vi) B/w any two different real nos. there lie infinitely many rational nos

(vii) Rationall are dense, B/w any two different rational no.s there lie infinitely many rationall nos,

(viii) B/w any two different rational no there lie infinitely many irrational no.

(ix) B/w any two different rational nos there lie infinitely many irrational nos

(8) define neighbourhood of a point.

Ans:- A set A(R) is called a neighbourhood of a point $\xi \in$ (zela) if there exists an open interval (a,b) containing zela and is contained in A

i.e $\xi \in (a,b) \subseteq A$

if ξ belong to A then $\epsilon = \min\{\xi - a, b - \xi\} > 0$

$(\xi - \epsilon, \xi + \epsilon) \subseteq (a,b)$

therefor if A is a neighbourhood of ξ then if contain of the points at a distance ϵ from ξ

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e.g; (i) Every open interval (a, b) is a neighbourhood of each of its point.

(ii) The interval [a, b) is a neighbourhood of each point of (a, b) but is not a neighbourhood of a because for a neighbourhood of a it has to contain points on the left of a also.

Similary, the interval (a, b] is a neighbourhood of each point of (a, b) but is not a neighbourhood of b.

✱ Define Interval

A subset I of R having atleast two elements is called an interval if whenever a, b belong to I and $a \leq x \leq b$ then x belongs of I.

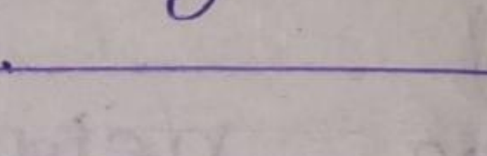
Types of interval

(i) closed interval :- A set $\{x : a \leq x \leq b\}$ where $a, b \in R$ & $a < b$ is said to be closed interval.

it is ~~defind~~ denoted by [a, b]

i.e $x \in [a, b]$

$\Rightarrow a \leq x \leq b$

eg, $x \in [1, 5]$ 

$\Rightarrow 1 \leq x \leq 5$

$\therefore x = 1, 2, 3, 4, 5$

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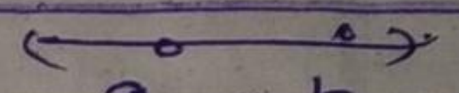
(ii) open internal :- A set $\{x : a < x < b\}$ where $a, b \in R$ & $a < b$ is said to be an open internal.

It is denoted by (a, b) or $]a, b[$

i.e $x \in]a, b[$

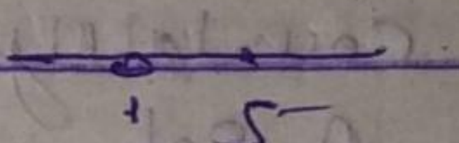
$\Rightarrow a < x < b$

e.g $x \in]1, 5[$



$\Rightarrow 1 < x < 5$

$x = 2, 3, 4$

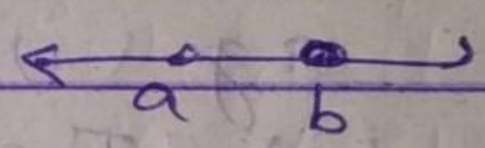


(iii) left half open interval :- A set $\{x : a < x \leq b\}$ where $a, b \in R$ & $a < b$ is said to be left half open interval,

It is denoted by $]a, b]$

i.e $x \in]a, b]$

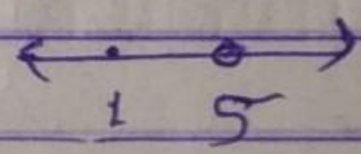
$\Rightarrow a < x \leq b$



eg, $x \in]1, 5]$

$\Rightarrow 1 < x \leq 5$

$x = 2, 3, 4, 5$



(iv) Right half open internal :-

A set $\{x : a \leq x < b\}$ where $a, b \in R$ & $a < b$ is said to be an right half open internal

It is denoted by $[a, b[$ or $[a, b)$

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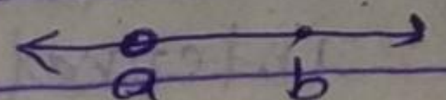
i.e $x_1 \in [a, b[$

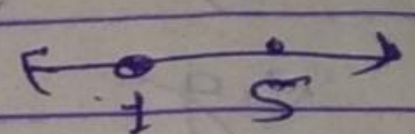
$\Rightarrow a \leq x < b$



eg $x \in [1, 5[$

$\Rightarrow 1 \leq x < 5$

$x = 1, 2, 3, 4$



* Define denumerable or Enumerable or countably infinite set.

→ A set x is said to be enumerable if there exists a 1-1 & on to fun from the set N to the set x.

* Define countable set with example

A set x is said to be countable if it is either finite or enumerable

eg (i) ϕ is countable

(ii) The set $\{1, 5, 9, 20\}$ is countable

(iii) The sets N, Z & ϕ are countable.

(8) Define Uncountable set

A set which is not countable is said to be uncountable

eg (i) The set R is uncountable

(ii) The set of irrational no is uncountable

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~~2016-19~~

1. Answer all questions:

(a) Define a bounded set

(b) ~~Every~~ bounded set has a.l.u.b

D'Alembert's ratio test

Statement :- A series of +ve terms Σu_n is convergent or divergent according as $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l < 1$ or > 1 and test fails for $l = 1$.

proof :- If is given that Σu_n is series of +ve terms and $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$ then for every $\epsilon > 0$ ∃ a +ve integer m such that

$$\left| \frac{u_{n+1}}{u_n} - l \right| < \epsilon \quad \text{from } n \geq m$$

$$l - \epsilon < \left| \frac{u_{n+1}}{u_n} \right| < l + \epsilon \text{ for } n \geq m \quad \text{--- (a)}$$

Case (1) If $l < 1$

$l + \epsilon < 1$

Puting $l + \epsilon = K < 1$

We have from (a)

$$\frac{u_{n+1}}{u_n} < l + \epsilon \text{ for } n \geq m$$

$$\frac{u_{n+1}}{u_n} < K \text{ for } n \geq m$$

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$\therefore u_{n+1} < k u_n$ for $n \geq m$

Putting $n = m, m+1, m+2$ ——

$u_{m+1} < k u_m$

$u_{m+2} < k u_{m+1} < k^2 u_m$

$u_{m+3} < k u_{m+2} < k^3 u_m$

— — — — — — — — —

— — — — —

— — — — —

$\therefore u_m + u_{m+1} + u_{m+2} + \cdots < u_m(1 + k + k^2 + k^3 + \cdots)$

this is in G.P whose

c.r $= k < 1$ so

this is a convergent Sequence

then by comparison test $\sum u_n$ is also convergent series

Case (II) If $\lambda > 1$

We can choose $\epsilon > 0$ s.t

$1 - \epsilon > 1$

putting $1 - \epsilon = \lambda > 1$

We have from (a)

$\frac{u_{n+1}}{u_n} > 1 - \epsilon$ for $n \geq m$

$\frac{u_{n+1}}{u_n} > \lambda$ for $n \geq m$

putting $n = m, m+1, m+2, m+3$

$u_{m+1} > \lambda u_m$

$u_{m+2} > \lambda u_{m+1} > \lambda^2 u_m$

$u_{m+3} > \lambda u_{m+2} > \lambda^3 u_m$

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$\therefore u_m + u_{m+1} + u_{m+2} \longrightarrow u_m(1+\lambda+\lambda^2+\lambda^3 \longrightarrow)$

this is in G.P whose ~~$(r = \lambda)$~~ $r = \lambda$ i.e. $\lambda > 1$ so this is a ~~diff~~ divergent series

then by comparison test Σu_n is also divergent.

case iii) If $\lambda = 1$

the test fails this can be shown by an example

let $\Sigma \frac{1}{n}$ & $\Sigma \frac{1}{n^2}$ be the two series

$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$ for both series

$\Sigma \frac{1}{n}$ & $\Sigma \frac{1}{n^2}$ but 1st is divergent but 2nd is convergent

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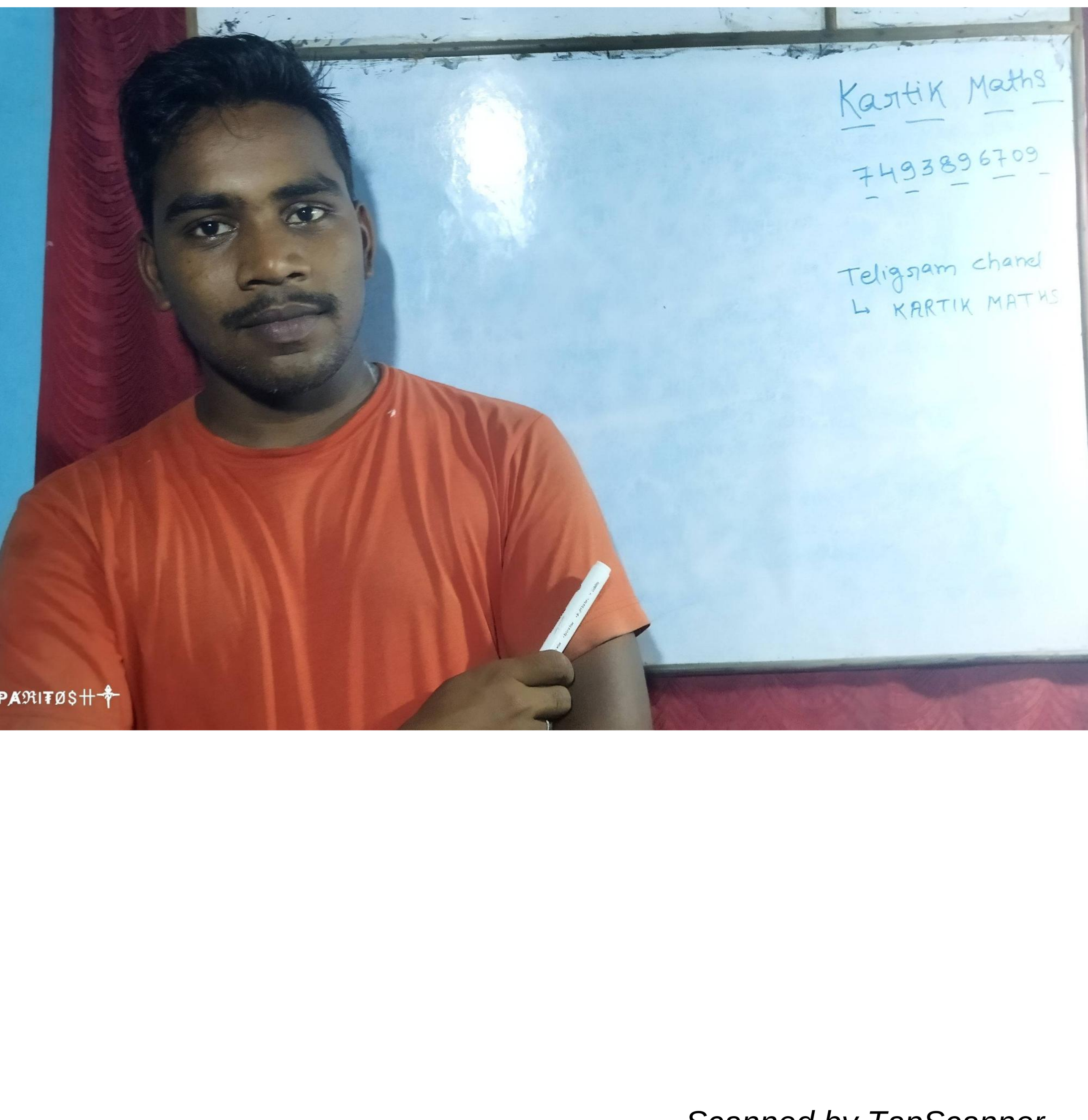

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